

# Level Set Methods for Variational Problems and Applications

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## Abstract

We formulate de-convolution, inverse interface, free boundary, obstacle problem and shape optimization problems as a variational problem with respect to a class of admissible interfaces. The interface is represented by the zero level set of a level set function. The unknown interface is seek by deforming the interface by Hamilton-Jacobi equation with a pre-scribed normal velocity. The normal vector is often chosen as the negative shape gradient of the costfunctional with respect to the interface. We discuss pre-conditioning techniques and Gauss-Newton method for selecting the normal velocity. A numerical result for the obstacle problem is presented.

## 1 Introduction

In this paper we discuss the application of the level set method for inverse interface problems and shape optimization. The interface and shape are represented by the zero level set of the level set function. Level set methods (e.g., [10],[13] and the references therein) are an extremely versatile tool for representing moving fronts in a variety of physical process, involving flow phenomena, crystal growth and phase transition among others.

The inverse interface problems can be formulated as a minimization of least-square data-fit-criterion over a class of the unknown interface. The shape optimization involves minimizing certain performance index over a class of admissible boundary shapes. In general we consider the minimization of the form

$$(1.1) \quad \min J(u, \Gamma)$$

subject to the constraint

$$(1.2) \quad E(u, \Gamma) = 0 \quad \text{in } Y$$

over a class  $Q_{ad}$  of admissible interfaces  $\Gamma$ . In general the state function  $u \in X$ , a Hilbert space is a function of  $\Gamma$ , denoted by  $u = u(\Gamma)$  which (uniquely) solves the constraint

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$E(u, \Gamma) = 0$  in a Hilbert space  $Y$  for given  $\Gamma$ . Thus, (1.1)-(1.2) reduces to the unconstrained minimization  $\min J(u(\Gamma), \Gamma)$  over  $\Gamma \in Q_{ad}$ . Or, we formally introduce the Lagrangian

$$(1.3) \quad L(u, p, \Gamma) = J(u, \Gamma) + \langle p, E(u, \Gamma) \rangle$$

where  $p \in Y^*$ , a Lagrange multiplier corresponding to the constraint, satisfies the adjoint equation. In general  $Y$  can also depend on  $\Gamma$ .

Our approach is based on

(a) the interface and shape are represented by the zero level set of the level set function  $\phi(t, x)$ , i.e.,

$$(1.4) \quad \Gamma_t = \{x : \phi(t, x) = 0\},$$

(b) the deformation speed of the level set is calculated as the gradient  $J'$  of the cost functional with respect to the shape, i.e., let  $\Gamma_{t+\Delta t}$  be the family of curves given by

$$(1.5) \quad \Gamma_{t+\Delta t} = \Gamma + \Delta t \vec{h}(x) \equiv \{x + \Delta t \vec{h}(x) \in \Omega : x \in \Gamma_t\}, \quad \Delta t > 0$$

and  $E(u_t, \Gamma_t) = 0$  and  $E(u_{t+\Delta t}, \Gamma_{t+\Delta t}) = 0$ , then the shape gradient  $J'$  is given by

$$(1.6) \quad (J', \nu \cdot \vec{h})_{\Gamma_t} = \frac{d}{dt} J(u_t, \Gamma_t) = \lim_{\Delta t \rightarrow 0^+} \frac{J(u_{t+\Delta t}, \Gamma_{t+\Delta t}) - J(u_t, \Gamma_t)}{\Delta t}.$$

in the direction  $\vec{h}(x)$  at  $\Gamma_t$ , where  $\nu$  is the outward normal at  $\Gamma_t$ , and

(c) the level set function  $\phi$  is evolved in time by the corresponding Hamilton-Jacobi equation [10];

$$(1.7) \quad \phi_t(t, x) + V(t, x) |\nabla \phi(t, x)| = 0,$$

where  $V(t, x)$  is an extension of  $J'(x)$  from  $\Gamma_t$  into  $R^d$ . That is, the motion of the level-set function is governed by

$$(1.8) \quad \phi_t + (V(t, x) \nu) \cdot \nabla \phi = 0,$$

where  $V(t, x) \nu$  is the velocity of the level curves and since  $\nu = \nabla \phi(t, x) / |\nabla \phi(t, x)|$ , it derives (1.7).

The normal deformation velocity  $V$  in (1.7) can be selected as a Newton (pseudo-Newton) direction of the cost functional  $J(u(\Gamma), \Gamma)$  or a Gauss-Newton direction in the least square cost as often being employed in general optimization problems and we discuss the details in Section 4.

In Section 2 we introduce the motivated examples including de-convolution problems and structural optimization as well as free boundary and obstacle problems. In Section 3 we discuss the shape gradient of the costfunction based on the adjoint equation method. In Section 4 we discuss a pre-conditioning technique and Gauss-Newton update. In Section 5 we discuss the numerical results for the obstacle problem.

In some of examples discussed in Section 2 the shape gradient involves the normal derivative of the solution as well as the adjoint variable along the interface  $\Gamma_t$ . As a consequence, in order to guarantee a numerically accurate shape gradient, the normal derivatives along the interface has to be calculated with high accuracy. In the joint work with Z. Li (e.g., [8]) we developed the second order accurate numerical discretization based on the so-called immersed interface method and it is used in our numerical implementation.

There has been increasing researches in applications of the level set method for inverse and optimization problems. In the context of inverse problems Santosa and co-workers on [9] applied level set techniques to inverse scattering problems and in [6] applied to electrical impedance tomography problems. In [14] and [11] the level-set equation approach is used for structural optimization problems.

## 2 Motivated Examples

In this section we discuss some examples that motivate our study.

### 2.1 De-convolution

Inverse scattering problems can be formulates as

$$z(x) = \int_{\Omega_0} K(x, y) u(y) dy + \text{Noise}$$

where  $z(x)$  and  $u(x)$  represent the distributed measurement and image defined on an open bounded domain  $\Omega_0$ , and  $K$  is the symmetric positive scattering kernel. We assume that the image  $u$  is binary, i.e., the image  $u$  is represented by

$$u(t, x) = \begin{cases} 1, & x \in \Omega_t^+ \\ 0, & x \in \Omega_t^- \end{cases}$$

We formulate the inverse scattering problem, reconstructing the image  $u$  from the observation  $z$  as a variational problem

$$\min J(\Gamma) = \int_{\Omega_0} \frac{1}{2} \left| \int_{\Omega_0} K(x, y) u(y) dy - z(x) \right|^2 dx.$$

By formula (3.4) it can be shown that

$$\left. \frac{d}{dt} J(\Gamma_t)(h) \right|_{t=0} = - \int_{\Gamma_t} \left[ \int_{\Omega_0} \left( \int_{\Omega} K(x, y) u(y) dy - z(x) \right) G(x, s) dx \right] \nu \cdot \vec{h}(s) ds.$$

Thus the maximum decent deformation direction of  $J$  is

$$V(t, x) = \int_{\Omega_0} \left( \int_{\Omega_0} K(x, y) u(y) dy - z(x) \right) G(x, s) dx.$$

The evaluation of  $V(t, x)$  can be performed efficiently, i.e., it also consists of the summation of the kernel over  $\Omega_t^+$  for the error

$$e(x) = \int_{\Omega_t^+} K(x, y) dy - z(x)$$

and then the weighted sum of the error

$$\int_{\Omega_0} e(x)G(x, s) dx$$

at  $s \in \Gamma_t$ .

## 2.2 Electrical Impedance Tomography

First we describe an inverse interface problem [6] for a problem motivated by electrical impedance tomography. Let  $\phi = \phi(t, x)$ ,  $t \geq 0$ ,  $x \in R^2$  denote the level set function. It defines a family of interfaces  $\Gamma_t$

$$\Gamma_t = \{x \in R^2 : \phi(t, x) = 0\}$$

and domains

$$\Omega_t^+ = \{x \in \Omega : \phi(t, x) > 0\}$$

$$\Omega_t^- = \{x \in \Omega : \phi(t, x) < 0\}.$$

We consider the following interface problem. Let  $\Omega_0 = (-1, 1)^2$ . The potential function  $u \in H^1(\Omega_0)$  satisfies

$$(2.1) \quad -div(\mu_t(x) grad u_t) = 0$$

with boundary condition

$$\frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega$$

where  $\partial\Omega_0$  is the boundary of  $\Omega_0$ . The conductivity  $\mu_t$  is piecewise constant and given by

$$(2.2) \quad \mu_t(x) = \begin{cases} \mu^+, & x \in \Omega_t^+ \\ \mu^-, & x \in \Omega_t^- \end{cases}$$

The domain  $\Omega^-$  represents the inhomogeneity of the conducting medium. That is, if  $\Omega^- = \emptyset$ , then we have a homogeneous conduction medium. We assume that  $\Omega^-$  is a finite union of simply connected open sets in  $\Omega$  and  $\Gamma$ , the union of  $C^2$  closed curves in  $\Omega$  represents the interface between the two open domains  $\Omega^+$  and  $\Omega^-$ . Let  $\hat{\Omega}$  be the region of the observation defined by

$$\hat{\Omega} = \{x \in \Omega_0 : dist(x, \partial\Omega_0) < 0.2\}$$

and we observe the potential function  $u$  on  $\hat{\Omega}$ . The data can be considered to be obtained, for example, from boundary measurements by numerical extension into the interior of  $\Omega_0$ .

We consider the inverse problem of identifying the unknown interface  $\Gamma$  from the observation  $z$  of  $u$  at  $\hat{\Omega}$ . Given the interface  $\Gamma$  let  $u(\Gamma) \in H^1(\Omega_0)/R$  denote the solution to the boundary value problem (1.1). We formulate the least square problem

$$(2.3) \quad \min \quad J(\Gamma) = \int_{\hat{\Omega}} \frac{1}{2} |u(\Gamma) - u|^2 dx + \epsilon \int_{\Gamma} 1 ds$$

over  $\Gamma \in Q_{ad}$ , where  $Q_{ad}$  is an admissible class of the interfaces  $\Gamma$ . The second term represents the perimeter regularization and  $\epsilon \geq 0$  is the Tichnov regularization parameter. If we assume  $\mu^- = \infty$  and  $\Omega^-$  consists only of one connected component, then the boundary value problem (1.1) reduces into

$$(2.4) \quad -\Delta u = 0 \quad \text{in } x \in \Omega^+$$

with boundary conditions

$$u = 0 \quad \text{on } \Gamma \quad \text{and} \quad \frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega^+.$$

### 2.3 Shape Optimization

Next we describe a shape optimization problem arising in the magnetic shaping of liquid metals [4].

$$(2.5) \quad \min \quad J(\Gamma) = \int_{\Omega} |\nabla u|^2 dx \quad \text{over } \Gamma$$

subject to

$$(2.6) \quad -\Delta u = j, \quad u = 0 \quad \text{on } \partial\Omega$$

and

$$(2.7) \quad \text{vol}(\Omega) \geq \bar{V},$$

where  $j$  is a smooth function with compact support describing the distribution of electrical current. In this case we set  $\Gamma = \partial\Omega$ , the boundary of simply connected domain  $\Omega$  and the solution  $u(\Gamma) \in H_0^1(\Omega)$  to (2.6) is a function of the boundary shape  $\Gamma$ .

### 2.4 Free Boundary Problem

We consider the Alt-Caffarelli problem [1], i.e., the problem of finding  $\Gamma$ , the free boundary such that

$$(2.8) \quad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 1 & \text{in } \Gamma_0 \\ u = 0 \text{ and } \frac{\partial u}{\partial \nu} = \lambda & \text{at } \Gamma \end{cases}$$

where  $\partial\Omega$  is the disjoint union of open and closed set  $\Gamma_0$  and  $\Gamma$ , and the boundary  $\Gamma_0$  and  $\lambda$  are known. It is known that the solution to (2.8) is a critical point of the following energy

$$(2.9) \quad E(u, \Gamma) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + |\lambda|^2) dx.$$

subject to

$$(2.10) \quad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 1 & \text{on } \Gamma_0 \text{ and } u = 0 \text{ on } \Gamma \end{cases}$$

In fact, it will be shown that

$$\left. \frac{d}{dt} E(u(\Gamma), \Gamma) \right|_{t=0} = \int_{\Gamma} \frac{1}{2} (-|\nabla u|^2 + |\lambda|^2) (\nu \cdot \vec{h}) ds$$

and thus the necessary optimality is given by

$$-|\frac{\partial u}{\partial \nu}|^2 + |\lambda|^2 = 0 \text{ on } \Gamma$$

## 2.5 Obstacle Problem

We consider the obstacle problem (e.g., see [5] and references therein)

$$(2.11) \quad \min \int_{\Omega_0} \left( \frac{1}{2} |\nabla u|^2 - u(x)f(x) \right) dx \quad \text{subject to } u(x) \leq \psi(x) \text{ a.e. in } \Omega$$

over  $u \in H_0^1(\Omega_0)$ . The necessary and sufficient optimality condition is given by

$$-\Delta u + \lambda = f, \quad \lambda = \max(0, \lambda + (u - \psi))$$

where  $\lambda \geq 0$  in  $L^2(\Omega)$  is the Lagrange multiplier. Let  $\Omega = \{x \in \Omega_0 : u(x) < \psi(x)\}$ . Then  $\lambda(x) = 0$  in  $\Omega$  and thus  $-\Delta u = f$  in  $\Omega$ .  $\Omega^c = \{x \in \Omega_0 : u(x) = \psi(x)\}$  is the contact region and we let  $\Gamma$  be the interface. If  $f \in L^2(\Omega_0)$  and  $\max(0, f - \Delta\psi) \in L^2(\Omega_0)$ , then it can be shown (e.g., [5]) that  $u \in H^2(\Omega_0)$ . Thus  $[\frac{\partial(u-\psi)}{\partial \nu}] = 0$  and  $u = \psi$  at  $\Gamma$ . It is a free boundary problem.

We now formulate the variational problem

$$(2.12) \quad \min \int_{\Omega} |\nabla(u - \psi)^+|^2 dx$$

subject to

$$(2.13) \quad -\Delta u = f \quad \text{in } \Omega, \quad u = \psi \text{ at } \Gamma = \partial\Omega.$$

### 3 Shape Gradient and Adjoint Equations

In this section we discuss the shape derivative of the costfunctional  $J(\Gamma)$  with respect to the shape  $\Gamma$ . Let  $\Gamma \in Q_{ad}$  be fixed and for  $|t|$  sufficiently small, let  $\Omega_t = F_t(\Omega)$  be the image of  $\Omega$  obtained by the mapping  $F_t : R^2 \rightarrow R^2$  defined as

$$(3.1) \quad F_t(x_1, x_2) = (x_1, x_2) + t \vec{h}(x_1, x_2).$$

For  $\varphi \in H^1(\Omega)$  and  $\varphi_t(\Omega_t)$  the material derivative of  $\varphi$  for field  $h \in (H^1(\Omega))^2$  is given by

$$(3.2) \quad \dot{\varphi}(x) = \lim_{t \rightarrow 0} \frac{\varphi_t(x + th) - \varphi(x)}{t} \quad \text{for } x \in \Omega^+.$$

If  $\varphi_t$  has a regular extension to a neighborhood of  $\bar{\Omega}_t$ , then

$$(3.3) \quad \varphi'(x) = \lim_{t \rightarrow 0} \frac{\varphi_t(x) - \varphi(x)}{t} = \dot{\varphi}(x) - h(x) \cdot \nabla \varphi(x), \quad x \in \Omega$$

is called the shape derivative of  $\varphi$ . These notations are standard in the theory of shape optimization, for example in [15] and references therein. Note that

$$(3.4) \quad \frac{d}{dt} \left( \int_{\Omega_t} \varphi_t dx_t \right) \Big|_{t=0} = \int_{\Omega} \dot{\varphi} + \varphi \operatorname{div} h dx = \int_{\Omega} \varphi' + \operatorname{div}(h\varphi) dx.$$

Assume that the shape derivative  $u'$  of  $u_t$  exists. If  $F(\Gamma) = J(u(\Gamma), \Gamma)$ , then

$$\frac{d}{dt} F(\Gamma) \Big|_{t=0} = \langle F'(u), u' \rangle + \lim_{t \rightarrow 0} \frac{F(u, \Gamma_t) - F(u, \Gamma)}{t}.$$

For the magnet shaping problem (2.5)–(2.7) we have

$$\begin{cases} -\Delta u' = 0 & \text{in } \Omega \\ u' + \vec{h} \cdot \nabla u = 0 & \text{at } \Gamma \end{cases}$$

Thus

$$(3.5) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega_t} |\nabla u_t|^2 dx \Big|_{t=0} \\ &= 2 \int_{\Omega} \nabla u' \cdot \nabla u dx + \int_{\Gamma} |\nabla u|^2 (\nu \cdot \vec{h}) ds = \int_{\Gamma} |\nabla u|^2 (\nu \cdot \vec{h}) ds, \end{aligned}$$

since

$$\int_{\Omega} \nabla u' \cdot \nabla u dx = \int_{\Omega} u \Delta u' dx = 0.$$

For the free boundary problem (2.10)–(2.11)

$$\begin{cases} -\Delta u' = 0 & \text{in } \Omega \\ u' + \vec{h} \cdot \nabla u = 0 & \text{at } \Gamma \\ u' = 0 & \text{at } \Gamma_0 \end{cases}$$

Thus

$$\begin{aligned} (3.6) \quad & \frac{d}{dt} \int_{\Omega_t} \frac{1}{2} (|\nabla u_t|^2 + |\lambda|^2) dx \Big|_{t=0} \\ &= \int_{\Omega} \nabla u' \cdot \nabla u \, dx + \int_{\Gamma} \frac{1}{2} (|\nabla u|^2 + |\lambda|^2) (\nu \cdot \vec{h}) \, ds \\ &= \int_{\Gamma} \frac{1}{2} \left( -\left| \frac{\partial u}{\partial \nu} \right|^2 + |\lambda|^2 \right) (\nu \cdot \vec{h}) \, ds, \end{aligned}$$

since

$$\int_{\Omega} \nabla u' \cdot \nabla u \, dx = \int_{\Omega} u' \Delta u \, dx + \int_{\Gamma} \frac{\partial u}{\partial \nu} u' \, ds = - \int_{\Gamma} \left| \frac{\partial u}{\partial \nu} \right|^2 (\nu \cdot \vec{h}) \, ds.$$

For the obstacle problem (2.12)–(2.13)

$$\begin{cases} -\Delta u' = 0 & \text{in } \Omega \\ u' + \vec{h} \cdot \nabla (u - \psi) = 0 & \text{at } \Gamma. \end{cases}$$

Thus

$$\begin{aligned} (3.7) \quad & \frac{d}{dt} \int_{\Omega_t} |\nabla (u_t - \psi)^+|^2 dx \Big|_{t=0} \\ &= 2 \int_{\Omega} \nabla (u - \psi)^+ \cdot \nabla u' \, dx + \int_{\Gamma} |\nabla (u - \psi)^+|^2 (\nu \cdot \vec{h}) \, ds. \\ &= \int_{\Gamma} |\nabla (u - \psi)^+|^2 (\nu \cdot \vec{h}) \, ds = \int_{\Gamma} \left| \left( \frac{\partial (u - \psi)}{\partial \nu} \right)^+ \right|^2 (\nu \cdot \vec{h}) \, ds, \end{aligned}$$

since

$$\int_{\Omega} \nabla (u - \psi)^+ \cdot \nabla u \, dx = 0.$$

Note that the shape derivative  $u'$  is only used to carry out the chain rule in above but the shape derivative of the costfunctional is not expressed in terms of  $u'$ . In fact it is possible to

evaluate the shape derivative of the costfunctional directly by the adjoint equation method. We present the arguments for problem (2.1)–(2.3) in what follows and it will be shown that

$$V(t, x) = (\mu^+ - \mu^-)([\nabla u_t]^+ \cdot [\nabla p_t]^-, \nu \cdot \vec{h})_{\Gamma_t}$$

where  $u_t, p_t \in H^1(\Omega_0)/R$  satisfy

$$\langle \mu_t \nabla u_t, \nabla \psi \rangle - (g, \psi)_{\Gamma_0} = 0$$

for all  $\psi \in H^1(\Omega_0)/R$  and

$$(\mu_t \nabla p_t, \nabla \psi)_{\Omega_0} - (\chi_{\hat{\Omega}}(u - z), \psi)_{\Omega_0} = 0$$

for all  $\psi \in H^1(\Omega)/R$ .

**Theorem 3.1** *The shape gradient of  $J(\Gamma_t) = \frac{1}{2} \int_{\hat{\Omega}} |u_t - z|^2 dx$  at  $t = 0$  is given by*

$$(3.8) \quad J'(\nu \cdot \vec{h}) = (\mu^+ - \mu^-)([\nabla u]^+ \cdot [\nabla p]^-, \nu \cdot \vec{h})_{\Gamma}$$

where  $p \in H^1(\Omega_0)/R$  satisfies

$$(3.9) \quad (\mu \nabla p, \nabla \psi)_{\Omega_0} - (\chi_{\hat{\Omega}}(u - z), \psi)_{\Omega_0} = 0$$

for all  $\psi \in H^1(\Omega_0)/R$ .

**Proof:** First note that

$$(\mu_t \nabla(u_t - u), \nabla \psi)_{\Omega_0} = ((\mu_t - \mu) \nabla u, \nabla \psi)_{\Omega_0}$$

for  $\psi \in H^1(\Omega_0)/R$ . Setting  $\psi = u_t - u$  in this,

$$|\nabla(u_t - u)| \leq \frac{1}{\mu_+} |(\mu_t - \mu) \nabla u| \leq M t^{1/2}.$$

if  $\xi_t = t^{-1/2}(u_t - u)$ , then there exists a subsequence (denoted by the same) such that  $\nabla \xi_t$  converges weakly to  $\xi$  in  $H^1(\Omega_0)/R$  as  $t \rightarrow 0^+$ . Since

$$(\mu_t \nabla \xi_t, \nabla \psi)_{\Omega_0} = (t^{-1/2}(\mu_t - \mu) \nabla u, \nabla \psi)_{\Omega_0},$$

thus for  $\psi \in C^1(\Omega_0)$  letting  $t \rightarrow 0^+$ , we obtain

$$\langle \mu \nabla \xi, \nabla \psi \rangle = 0$$

and thus  $\xi = 0$ . Since  $H^1(\Omega_0)$  is compactly embedded into  $L^2(\Omega_0)$ , thus

$$\lim_{t \rightarrow 0^+} \frac{|u_t - u|_{L^2(\Omega_0)}}{t^{1/2}} = 0.$$

Note that

$$(3.10) \quad J_0(u_t) - J_0(u) = \int_{\hat{\Omega}} \left\{ (u - z, u_t - u) + \frac{1}{2} |u_t - u|^2 \right\} dx$$

and

$$(3.11) \quad (\chi_{\hat{\Omega}}(u - z), u_t - u) = (\mu \nabla(u_t - u), \nabla p)_{\Omega_0} = -((\mu_t - \mu) \nabla u_t, \nabla p)_{\Omega_0}$$

where we used  $(\mu_t \nabla u_t, \nabla p)_{\Omega_0} = (\mu \nabla u, \nabla p)_{\Omega_0}$ .

We represent the solution  $u_t$  by the single-layer potential

$$u_t(x) = \frac{1}{2\pi} \int_{\Gamma_t} G(x, y) \phi_t(y) dy + \frac{1}{2\pi} \int_{\Gamma_0} G(x, y) \phi_0(y) dy$$

where  $G$  is the Green's kernel function ( $G(x, y) = \log(|x - y|)$  in  $R^2$ ). We determine  $(\phi_t, \phi_0)$  so that the boundary condition and the flux continuity are satisfied. From the potential limiting theory [2], we have

$$\frac{\partial}{\partial \nu} u_t^\pm(x) = \frac{1}{2\pi} \int_{\Gamma_t} \frac{\partial}{\partial \nu_x} G(x, y) \phi_t(y) dy \pm \frac{\phi_t(x)}{2}$$

at  $\Gamma_t$ . Thus,  $(\phi_t, \phi_0)$  satisfy the Fredholm integral equation of the second kind;

$$\left\{ \begin{array}{l} \frac{\mu^+ + \mu^-}{2} \phi_t(x) + \frac{\mu^+ - \mu^-}{2\pi} \int_{\Gamma_t} \frac{\partial}{\partial \nu_x} G(x, y) \phi_t(y) dy \\ \quad + \frac{\mu^+ - \mu^-}{2\pi} \int_{\Gamma_0} \frac{\partial}{\partial \nu_x} G(x, y) \phi_0(y) dy = 0, \quad x \in \Gamma_t \\ \\ \frac{1}{2} \phi_0(x) + \frac{1}{2\pi} \int_{\Gamma_t} \frac{\partial}{\partial \nu_x} G(x, y) \phi_t(y) dy \\ \quad + \frac{1}{2\pi} \int_{\Gamma_0} \frac{\partial}{\partial \nu_x} G(x, y) \phi_0(y) dy = g(x), \quad x \in \Gamma_0. \end{array} \right.$$

Let  $\Gamma_t$  be a  $C^2$  closed curve. By the Riesz-Fredholm theory there it has a unique solution  $(\phi_t, \phi_0) \in C^{0,\alpha}$ , the space of Hölder continuous functions with exponent  $0 < \alpha < 1$ , provided that  $g \in C^{0,\alpha}$ . It follows from [2] that  $u$  is continuous across  $\Gamma_t$  and piecewise  $C^{1,\alpha}$ . Moreover it can be proved that

$$\mu_t \nabla u_t \rightarrow \mu^- \nabla u \quad \text{in } x \in \Omega_t^+ \cap \Omega^-$$

$$\mu_t \nabla u_t \rightarrow \mu^+ \nabla u \quad \text{in } x \in \Omega_t^- \cap \Omega^+$$

as  $t \rightarrow 0$ . Thus (3.8) follows from (3.10)–(3.11).  $\square$

## 4 Gauss-Newton method and Pre-conditioning

Consider the minimization of the form

$$(4.1) \quad \min |Cu(\Gamma) - z|_Z^2$$

where  $u(\Gamma) \in X$  is the solution to the equality constraint  $E(u(\Gamma), \Gamma) = 0$  and  $C$  is a bounded linear operator from  $X$  into a Hilbert space  $Z$ . Assume that the shape derivative  $u'_t$  of  $u_t(x)$  exists. Define the Jacobian  $G : v = \nu \cdot \vec{h} \in L^2(\Gamma_t) \rightarrow Z$  of  $\Gamma \rightarrow Cu(\Gamma)$  by

$$G_tv = Cu'_t.$$

The Gauss-Newton step is given by

$$(4.2) \quad \min |Gv + Cu_t - z|_Z^2 \quad \text{over } v \in Q$$

where  $Q$  is an admissible class of normal deformations  $v$ . In applications  $C$  is either a compact operator or not injective and thus  $G$  can be highly singular. In order to deal with this lack of sensitivity we either parameterize the normal deformations  $v$  or consider the regularized problem

$$(4.3) \quad \min |Gv + Cu_t - z|_Z^2 + \beta |v|_Q^2$$

where  $|v|_Q$  is a regularization semi-norm on functions on  $\Gamma$ .

### 4.1 Pre-conditioning

It should be noted that the shape gradient are evaluated under a regular class of interfaces  $\Gamma$  and thus of deformation vectors  $\vec{h}$ . As discussed in [16] that in general the shape gradient is a distribution and it is necessary to precondition it by re-norming in the continuous case so that the gradient method converges. As in [16] we use the re-norming based on the shape Hessian of the cost functional. Despite the fact that the shape Hessian is usually not coercive in the norm for which the differentiability holds, it does define a norm such that the iterates remain in the defined normed space. In general the shape Hessian is not symmetric and definite. Thus we extract the nonnegative symmetric portion of the Hessian by omitting indefinite terms. We examine the two specific cases.

In the case (4.1) of the least square problem the renorming is given by the quadratic form

$$(4.4) \quad |Gv|_Z^2 + \beta (Hv, v)_\Gamma$$

where  $H$  is the Laplace Beltrami operator. It is motivated by the following fact. We consider the perimeter constraint

$$P(\Gamma_t) = \int_{\Gamma_t} ds$$

Then, the shape gradient of  $P$  is given by

$$P'(\nu \cdot \vec{h}) = \int_{\Gamma} \kappa (\nu \cdot \vec{h}) ds$$

and the shape Hessian of  $P$  is given by

$$P''(\phi, \psi) = \int_{\Gamma} (\nabla_{\Gamma} \phi \cdot \nabla_{\Gamma} \psi + \det D\nu \phi \psi) ds$$

for  $\phi = \nu \cdot \vec{h}$  and  $\psi = \nu \cdot \vec{k}$ , where  $\kappa = \nabla \cdot \nu$  is the mean-curvature,  $D\nu$  is the Jacobian of the normal vector  $\nu$  and  $\nabla_{\Gamma} \phi = \nabla \phi - \frac{\partial \phi}{\partial \nu} \nu$ . Here

$$(4.5) \quad (H\phi, \psi) = \int_{\Gamma} \nabla_{\Gamma} \phi \cdot \nabla_{\Gamma} \psi ds$$

is the definite part of the Hessian. Thus, (4.4)–(4.5) leads to the regularized Gauss-Newton direction

$$V = (G^*G + \epsilon H)^{-1}G^*(u - z)$$

where  $G^*(u - z) = J'(\Gamma)$ .

In the case shape optimization (2.5)–(2.7) the necessary optimality condition is given by

$$L'(u(\Gamma), \Gamma, \lambda) = |\nabla u|^2 - \lambda = 0$$

where  $\lambda \in \mathbb{R}$  is the Lagrange multiplier corresponding to the volume constraint  $\int_{\Omega} dx - \bar{V} = 0$  and the Lagrangian  $L$  is given by

$$L(u(\Gamma), \Gamma, \lambda) = \int_{\Omega} |\nabla u|^2 dx + \lambda (\bar{V} - \int_{\Omega} dx).$$

Note that

$$C(\Gamma_t) = L'(\nu \cdot \vec{h}) = \int_{\Omega_t} \nabla \cdot ( (|\nabla u_t|^2 - \lambda) \vec{h} ) dx$$

We evaluate the Hessian of  $L$  assuming the vector field  $\vec{h}$  is autonomous. That is, it is the shape gradient of  $C$  with respect to  $\Gamma$  in the direction of  $\vec{k}$  and by (3.4) is given by

$$C'(\nu \cdot \vec{k}) = 2 \int_{\Omega} \nabla \cdot ( (\nabla u' \cdot \nabla u) \vec{h} ) dx + \int_{\Gamma} \nabla \cdot ( (|\nabla u|^2 - \lambda) \vec{h} ) (\nu \cdot \vec{k}) ds,$$

where  $u'$  satisfies

$$-\Delta u' = 0 \text{ in } \Omega, \quad u' + \frac{\partial u}{\partial \nu} (\nu \cdot \vec{k}) = 0 \text{ on } \Gamma.$$

It thus can be shown as in [16] that at the minimizer  $|\nabla u|^2 = \lambda$

$$L''(\nu \cdot \vec{h}, \nu \cdot \vec{k}) = 2\lambda \langle (S + \kappa^2 I)(\nu \cdot \vec{k}), \nu \cdot \vec{h} \rangle$$

where  $S$  is the Derezlet-to-Neumann operator from  $H^{\frac{1}{2}}(\Gamma)$  into  $H^{-\frac{1}{2}}(\Gamma)$ , i.e.,

$$S(\nu \cdot \vec{k}) = \frac{\partial v}{\partial \nu}$$

$$-\Delta v = 0 \text{ in } \Omega, \quad v = (\nu \cdot \vec{k}) \text{ on } \Gamma.$$

Thus, the pseudo-Newton direction is given by

$$V = -(S + \alpha I)^{-1} L'(u, \lambda),$$

for some  $\alpha > 0$ , i.e.,  $V = -y$  on  $\Gamma$ , where  $y \in H^1(\Omega)$  satisfies

$$-\Delta y = 0 \text{ in } \Omega, \quad \alpha y + \frac{\partial y}{\partial \nu} = |\nabla u|^2 - \lambda \text{ on } \Gamma.$$

## 5 Numerical Result

The level set method can be summarized as

- Set an initial level set function  $\phi^0(x)$  as initial guess of the unknown shape  $\Gamma_0 = \{x \in \Omega_0 : \phi^0(x) = 0\}$ .
- Solve equation  $E(u_k, \Gamma_k) = 0$  for  $u_k = u(\Gamma_k)$ , (where we use  $k$  to indicate the quantities in the  $k$ -th step) and evaluate the normal deformation vector  $V_k$  at  $\Gamma_k$ .
- Extend the velocity  $V_k$  to a computational tube  $|\phi^k| \leq \delta$ , where  $\delta$  is the width of the tube.
- Update the level set function  $\phi^k$  by solving the Hamilton-Jacobi equation

$$\phi_t + V_k |\nabla \phi| = 0, \quad \phi(0, x) = \phi^k(x)$$

- Set  $\Gamma_{k+1} = \{x \in \Omega_0 : \phi^{k+1}(x) = 0\}$  and re-normalize  $\phi^{k+1}$  as the sign distance function from  $\Gamma_{k+1}$ .

We used the Gudnov-type scheme (e.g, [12]) for the HJ equation on a fixed Cartesian grids with uniform meshsize and time stepsize  $\Delta t > 0$  (satisfying CFL condition); i.e.,

$$\frac{\phi^{k+1} - \phi^k}{\Delta t} + V_k |\nabla \phi^k| = 0,$$

where  $|\nabla \phi^k|$  is evaluated using a WENO (Weighted Essential Non-oscillatory)-scheme [7]. The extension of velocity  $V_k$  can be carried out by an upwind scheme along the normal direction originated from the interface  $\Gamma_k$ ;

$$V_t + \text{sign}(\phi^k) \nabla V \cdot \frac{\nabla \phi^k}{|\nabla \phi^k|} = 0.$$

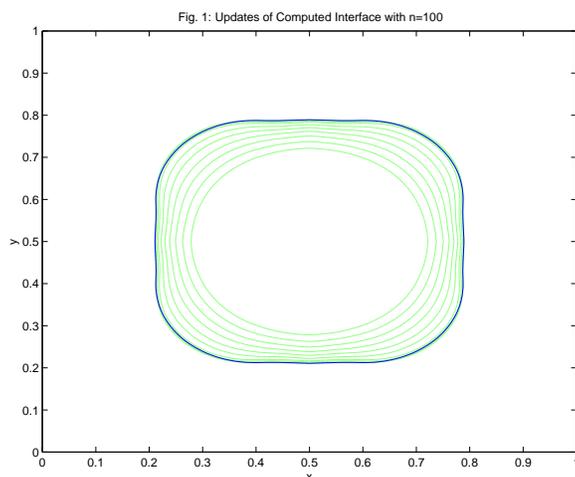
Re-initialization  $\phi^k$  as the signed distance function can be performed by solving the Eikonal equation

$$|\nabla\phi| = 1, \quad \phi(x) = 0 \text{ on } \Gamma_{k+1}.$$

We used the time-marching scheme [17] on the computational tube based on

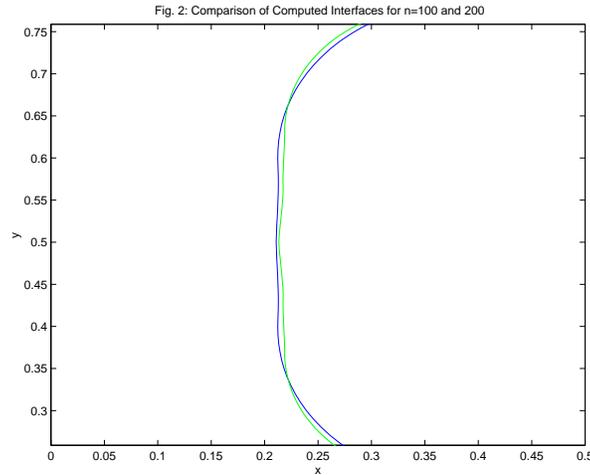
$$\phi_t + \text{sign}(\phi)(|\nabla\phi| - 1) = 0.$$

We tested the proposed algorithm for the obstacle problem (2.11)–(2.13). We set  $f = 50$  and  $\psi = 1$  on  $\Omega_0 = (0, 1) \times (0, 1)$ . We used the second order accurate numerical discretization of equation (2.13) on the Cartesian grid with uniform mesh size  $\Delta x = \Delta y = \frac{1}{n}$  based on the immersed interface method [8]. Successive updates  $\Gamma_k$  of computed interface are shown in Fig. 1 with iteration number 10 for the case  $n = 100$ . In Fig. 2 a comparison of computed interfaces with  $n = 100$  and  $n = 200$  is shown. The number of iterates for the case  $n = 200$  is 15. The contact region is the inside area enclosed by the interface  $\Gamma$ .



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