Fractional Evolution Equations and Applications

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Abstract

In recent years increasing interests and considerable researches have been given to the fractional differential equations both in time and space variables. These are due to the applications of the fractional differential operators to problems in a wide areas of physics and engineering science and a rapid development of the corresponding theory. Motivating examples include the so-called continuous time random walk process and the Levy process model for the mathematical finance. Basset integral is appearing in the equation of motion of a particle moving through a fluid. A fractional diffusion equation is derived as a homogenization of heterogeneous groundwater flow. In this lecture we develop solution methods based on the linear and nonlinear semigroup theory and apply it to solve the corresponding inverse and optimal control problems. The theory is applied to concrete examples including fractional diffusion equation, Navier-Stokes equations and conservation laws. For the linear case we develop the operator theoretic representation of solutions and the sectorial property of the fractional operator in time is used to establish the regularity and asymptotic of the solutions. The property and stability of the solutions as well as numerical integration methods are discussed. The lecture also covers the basic theory and application of the so-called Crandall-Ligget theory and the DS-approximation theory developed by Kobayashi-Kobayashi-Oharu for evolution operator and the semi-linear theory based on the sectorial estimates of the fractional equation.

1 Introduction

In this monograph we consider the fractional power equation of the form

$$D_t^\alpha u = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} u'(s)\,ds = Au(t) + f(t), \quad u(0) = x \tag{1.1}$$ fra

in a Banach space X, where A is an m-dissipative linear or nonlinear operator in X. Here, D_t^α is the Caputo fractional derivative of order α with $0 < \alpha < 1$. Our analysis

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will focus on fractional power equations in time but A may represent the fractional power operator in space such as fractional power Laplacean $-(-\Delta)^{\beta}$ (also, see Section). In general we consider equations of the form

$$\int_0^t g(t-s)u'(s)\,ds = Au(t) + f(t), \quad u(0) = x, \tag{1.2}$$

where we assume $g(s) \geq 0$ is monotonically deceasing and integrable on any finite interval $(0, R)$. Equation (1.1) is the special case of with

$$g(t) = g_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{1.3}$$

As shown in Appendix, such an equation arises in the continuous time random walk model for groundwater movement in naturally fractured and heterogeneous porous aquifers [3] and tick-by-tick dynamics of financial markets [5]. Boussinesq (1885) and Basset (1888) found that the force F on an accelerating spherical particle in a viscous fluid is given by

$$F = \frac{3}{2}D^2\sqrt{\pi\rho_c\mu_c}\int_0^t (\frac{\frac{Du}{Ds} - v'(s)}{\sqrt{t-s}}\,ds,$$

where D is the particle diameter, $\frac{D}{Ds}$ is the material derivative, and u and v are the fluid and particle velocity vectors, respectively. Thus, one can the Basset equation as (1): with $g = \delta_0 + k\frac{t^{-1/2}}{\Gamma(1/2)}$. i.e.

$$u'(t) + kD_t^{\frac{1}{2}}u = Au(t) + f(t), \quad u(0) = x.$$

Or, in general

$$g(t) = \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)}e^{-\beta t}\,d\mu(\alpha)$$

where μ is a positive measure on $(0, 1]$. The exponential decay model is

$$g(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}e^{-\beta t}$$

for some $\beta > 0$. Or, in general

$$g(t) = \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)}e^{-\beta t}\,d\tilde{\mu}(\beta)$$

where $\tilde{\mu}$ is a positive measure on $[0, \infty)$.

Since by the change of variable $t - s = -\theta$

$$\int_0^t g(t-s)u'(s)\,ds = \int_{-t}^0 g(-\theta)u'(t+\theta)\,d\theta,$$

we have

$$\int_0^t g(t-s)u'(s)\,dx = \frac{d}{dt}\int_0^t g(t-s)u(s)\,ds - g(t)u(0). = \frac{d}{dt}\int_{-\infty}^t g(t-s)u(s)\,ds, \tag{1.4}$$

where we set $u(s) = x,\ s \le 0$. Equivalently,

$$\int_0^t g(t-s)u'(s)\,dx = \frac{d}{dt}\int_0^t g(t-s)(u(s)-u(0))\,ds.$$

Thus, (1) is written as the fractional differential equation of the Riemann-Liouville form:

$$\frac{d}{dt}\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}u(s)\,ds = Au(t) + f(t) + \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}u(0),$$

or

$$\frac{d}{dt}\int_0^t g(t-s)(u(s)-u(0))\,ds = Au(t) + f(t).$$

There are several approaches to define the solution to (1.1). One uses the Mitag-Leffler function $E_{\alpha,\beta}(t)$ defined by

$$E_{\alpha,\beta}(\lambda t) = \sum_{n=0}^{\infty} \frac{\lambda^n t^{n\alpha}}{\Gamma(n\alpha+\beta)}. \tag{1.5}$$ ML

That is, $E_{\alpha,1}$ satisfies

$$D_t^\alpha E_{\alpha,1}(\lambda t) = \lambda\, E_{\alpha,1}(\lambda t).$$

Suppose A has a spectral resolution

$$A\phi = \int_C \lambda d\mathcal{E}(\lambda)\phi,$$

then it can be shown that for $f = 0$ the solution to (1.1) is given by

$$u(t) = \int_C E_{\alpha,1}(\lambda t)d\mathcal{E}(\lambda)x.$$

The other approach is the integral equation approach. Let J_t^α be the operator defined by

$$J_t^\alpha\phi = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s))\,ds.$$

Then, $J_t^\alpha D_t^\alpha\, u = u(t) - x, ; t \ge 0$ and we obtain the integral equation for u;

$$u(t) = x + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(Au(s) + f(s))\,ds,$$

Thus, one can apply the maximum monotone operator theory to establish the existence and regularity of solutions [1, 4].

Our approach is based on the following observation. One can write (1.1) as the functional differential equation

$$\int_{-\infty}^0 g(\theta)u'(t+\theta)\,d\theta = Au(t) + f(t), \quad \text{with initail value } u(\theta) = \phi(\theta),\ \ \theta \le 0. \tag{1.6}$$ fde

where $t - s = -\theta \le 0$ and $\theta \in R^- \to g(\theta) = g(-\theta)$ is the even extension of g and is monotonically increasing on R^-. Note that if we let $\phi(\theta) = x,\ \theta \le 0$ and

$g(\theta) = \frac{1}{\Gamma(1-\alpha)}|\theta|^{-\alpha}$, then (1.6) reduces to (1.1). That is, if $\theta \to u(t+\theta)$ is absolute continuous, it follows from (??) (see, Section 2.2 for the precise discussions) that

$$\int_{-\infty}^0 g(\theta)u'(t+\theta)\,d\theta = \int_0^t g(t-s)u'(s)\,ds.$$

We then embed the solution $u(t) = z(t,0)$ in the state (history) space

$$z(t,\theta) = u(t+\theta) \in Z = C((-\infty,0],X).$$

Then, (1) has the Markovian form as the evolution equation in Z:

$$\frac{d}{dt}z(t) = \mathcal{A}(t)z(t), \tag{1.7}$$

evol

where the operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t)\phi = \phi'(\theta), \quad \theta \in (-\infty,0] \tag{1.8}$$

evol0

in Z with domain

$$dom\,(\mathcal{A}(t)) = \{\phi \in Z : \phi' \in Z \text{ and } \int_{-\infty}^0 g(\theta)\phi'(\theta)\,d\theta = A\phi(0)+f(t), \quad \phi(0) \in dom(A)\}. \tag{1.9}$$

evol1

Dynamics (1) is embedded in (1.7) as the non-local boundary value condition as $\theta = 0^+$ for the first order differential operator $\mathcal{A}(t)$. We analyze the well-posedness and the property of of solutions to (1) based on the semigroup generated by (1.7), i.e., show that the solution map $(x,f) \in X \times C(0,T;X) \to u(t) \in C(0,T;X)$ exists and continuous. It will be shown that if A is dissipative and maximal monotone in X, then $\mathcal{A}(t)$ is dissipative and maximal monotone in Z. We then use the semigroup generation theory to define the solution $z(t) \in C(0,T;Z)$ to (1.7) and the solution to (1) by $u(t) = z(t,0) \in C(0,T;X)$. In this way we can define the solution to a more general class of equations of the form (??). In the case of (1.1) with a closed linear operator A, we have

$$u(t) = x + A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)\,ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds.$$

for all $x \in X$ and $f \in C(0,T;X)$, i.e.,

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)\,ds \in C(0,T;dom\,(A)).$$

For the case of a closed linear operator A, we also develop the operator theoretic approach to (1.1) and (1). If we take the Laplace transform of (1.1)

$$\lambda^\alpha \hat{u} = \lambda^{\alpha-1}x + \hat{f}$$

and thus

$$\hat{u} = (\lambda^\alpha I - A)^{-1}(\lambda^{\alpha-1}x + \hat{f})$$

assuming A is maximal monotone. Here, for $0 < \alpha < 1$ there exists $\theta > 0$ such that

$$|(\lambda^\alpha I - A)^{-1}| \le \frac{M_\alpha}{|\lambda|^\alpha} \quad \text{on the sector } \Sigma_\theta = \{z \in C : \arg(z) \le \frac{\pi}{2} + \theta\} \cap \{z \ne 0\}.$$

Thus, we have the representation of the solution operator $u(t) = S(t)x$:

$$S(t)x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda^\alpha\, I - A)^{-1}\lambda^{\alpha-1}x\, d\lambda.$$

where $\Gamma_{\gamma,\delta}$ be the integration path defined by

$$\Gamma^{\pm} = \{z \in C : |z| \geq \delta,\ arg(z) = \pm(\frac{\pi}{2}+\gamma)\}, \quad \Gamma_0 = \{z \in C : |z| = \delta,\ |arg(z)| \leq \frac{\pi}{2}+\gamma\}.$$

For $t > 0$ define $P(t) \in \mathcal{L}(X)$ by

$$P(t)x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda^\alpha\, I - A)^{-1}x\, d\lambda.$$

Then, we have the solution representation

$$u(t) = S(t)x + \int_0^t P(t-s)f(s)\, ds. \tag{1.10}$$ solr

We will analyze the solution properties based on this operator representation of $S(t)$ and $P(t)$ and establish the regularity and asymptotic property as $t \to \infty$ of $S(t)$ and $P(t)$. For the sectorial operator A we analyze the properties of $S(t)$ and $P(t)$ based on the fractional operator calculus.

For the semilinear equation with $Au = A_0u + F(u)$, we define the mild solution by the solution representation;

$$u(t) = S(t)x + \int_0^t P(t-s)(F(u(s) + f(s))\, ds. \tag{1.11}$$ semil

where $S(t)$, $P(t)$ are generated by A_0. We establish the existence of local and global solutions to (1.11) based on the properties $S(t)$ and $P(t)$ for the general case and the case when A_0 is a sectorial operator.

In summary the following is an outline of our presentation.

Plan of the Manuscript

[1] Well-posedness of (1.6) using C_0-semigroup theory for linear case, Section 2.

[2] Well-posedness of (1.6) using Crandall-Ligget theory for nonlinear monotone graph, Section 3.

[3] Evolution case $A = A(t)$ using the DS-approximation theory of Kobayashi-Kobayashi-Oharu, Section 4.

[4] Operator theoretic method and Sectorial Calculus based on the resolvent, Section 5.

[5] Solution Representations for Caputo equation and Riemann-Liouville equation, Section 5.

[6] Nonhomogeneous equations and Variation of constant formula, Section 6.

[7] Dual system, weak solutions and Control problems, Section 7

[8] Fractional wave equations $(0 < \alpha < 2)$, Section 8.

[9] Finite Difference approximation, stability and convergence analysis, Section 9.

[10] Local solutions to nonlinear fractional power ODEs, Section 10.

[11] Local solutions to semi-linear fractional power PDEs, Section 11.

[12] Examples, fractional diffusion, conservation law, Hamilto-Jacobi and Navier-Stokes equations, Section 12.

[13] Nonlocal and fractional PDEs in space, Section 13

[14] Eigenvalue Problems for fractional operator, Section 14

[15] CTRM model and Fractional diffusions via homogenization, Appendix.

2 Wellposedness for a closed linear operator A

In this section we consider the case A is m-dissipative in a Banach space X and show that $\mathcal{A}$ defined by (1.8)–(1.9) generates the C_0-semigroup on appropriate state spaces Z.

2.1 Weighted $Z = L_g(-\infty, 0; H)$ state space

First, we consider A is a closed linear operator in X. Let $X = H$ be a Hilbert space. Let $\mathcal{A}$ be the linear operator defined by

$$\mathcal{A}\phi = \phi'(\theta)$$

on the weighted history space $Z = L^2_g(-\infty, 0; H)$ with norm

$$|\phi|_Z^2 = \int_{-\infty}^0 g(\theta)\, |\phi(\theta)|_H^2 \, d\theta$$

with domain

$$dom\,(\mathcal{A}) = \{\phi \in Z : \phi' \in Z \text{ and } \int_{-\infty}^0 g(\theta)\phi'(\theta)\, d\theta = A\phi(0), \quad \phi(0) \in dom(A)\}.$$

The evolution equation (1) for u is embedded into the non-local boundary condition at $\theta = 0^+$ in the domain of the closed operator $\mathcal{A}$.

thm2.1 **Theorem 2.1.** *Assume A is m-dissipative. Then, $\mathcal{A}$ is m-dissipative, i.e.,*

$$R(\lambda\, I - \mathcal{A}) = Z \quad \text{for } \lambda > 0.$$

Thus, $\mathcal{A}$ generates the C_0-semigroup on Z.

Proof: Define

$$g_\epsilon(\theta) = \frac{1}{\epsilon} \int_{\theta-\epsilon}^{\theta} g(\theta)\, d\theta$$

For $\phi \in dom(\mathcal{A})$

$$(\mathcal{A}\phi, \phi)_Z = \int_{-\infty}^0 g(\theta)(\phi'(\theta), \phi(\theta) - \phi(0))_H \, d\theta + (\int_{-\infty}^0 g(\theta)\phi'(\theta)\, d\theta, \phi(0))_H,$$

where

$$\int_{-\infty}^{0} g(\theta)(\phi'(\theta),\phi(\theta)-\phi(0))_H = \lim_{\epsilon\to 0^+}\int_{-\infty}^{0} g_\epsilon(\theta)(\phi'(\theta),\phi(\theta)-\phi(0))_H\,d\theta$$
$$= -\frac{1}{2}\lim_{\epsilon\to 0^+}\int_{-\infty}^{0} g'_\epsilon(\theta)|\phi(\theta)-\phi(0)|^2\,d\theta \le 0, \tag{2.1}$$

sta0

since $g'_\epsilon(\theta)\ge 0$. Since

$$(\int_{-\infty}^{0} g(\theta)\phi'(\theta)\,ds,\phi(0))_H = (A\phi(0),\phi(0)) \le 0$$

we have $(\mathcal{A}\phi,\phi)_Z \le 0$. If we define for $\lambda > 0$

$$\psi(\theta) = \int_{\theta}^{0} e^{\lambda(\theta-\xi)} f(\xi)\,d\xi, \tag{2.2}$$

psi

then

$$\lambda\psi - \psi' = f \in Z, \quad \psi(0) = 0,$$

and $|\psi|_Z \le \dfrac{1}{\lambda}|f|_Z$. Let

$$I = \int_{-R}^{-\delta} g(\theta)\psi'(\theta)\,d\theta = g(-\delta)\psi(-\delta) - g(-R)\psi(-R) - \int_{-R}^{\delta} g'(\theta)\psi(\theta)\,d\theta. \tag{2.3}$$

bdd

Since for all $-\infty < R < \delta < 0$

$$|\psi(\theta)| \le \frac{1}{\sqrt{\lambda g(\theta)}}\,|f|_Z \quad \text{and} \quad \int_{-R}^{-\delta}\frac{(g')^2}{g}d\theta \text{ is bounded,}$$

it follows that for some $M > 0$

$$|\int_{-\infty}^{0} g(\theta)\psi'(\theta)\,d\theta| \le M\,|\psi|_Z.$$

Let

$$\Delta(\lambda) = \lambda\int_{-\infty}^{0} e^{\lambda\theta} g(\theta)\,d\theta. \tag{2.4}$$

Delta

Since A is m-dissipative, $\lambda\,\phi - \mathcal{A}\phi = f$ has the solution

$$\phi = (\lambda\,I - \mathcal{A})^{-1} f = e^{\lambda\theta}\phi(0) + \psi(\theta) \in \;\; dom(\mathcal{A}),$$

where

$$\phi(0) = (\Delta(\lambda)\,I - A)^{-1}\int_{-\infty}^{0} g'(\theta)\psi(\theta)\,d\theta.$$

The theorem now follows from the Lumer-Phillips theorem IK [2]. □

2.2 State space $Z = C((-\infty, 0]; H)$

Let $X = H$ be a Hilbert space and consider the state space

$$Z = C((-\infty, 0]; H).$$

Define a linear operator $\mathcal{A}$ by

$$\mathcal{A}\phi = \phi'$$

with domain

$$dom(\mathcal{A}) = \{\phi' \in Z : \phi(0) \in dom(A), \quad \int_{-\infty}^0 g(\theta)\phi'(\theta)\,d\theta = A\phi(0)\}$$

Then,

$$\begin{aligned}
&(\int_{-\infty}^0 g_\epsilon(\theta)\phi'(\theta)\,d\theta, \phi(0)) = (\int_{-\infty}^0 \frac{g(\theta)-g(\theta-\epsilon)}{\epsilon}\,(\phi(0)-\phi(\theta))\,d\theta, \phi(0)) \\
&= \int_{-\infty}^0 \frac{g(\theta)-g(\theta-\epsilon)}{\epsilon}((\phi(0),\phi(0)) - (\phi(\theta),\phi(0)))\,d\theta \qquad (2.5) \quad \boxed{\texttt{sta}} \\
&\geq \int_{-\infty}^0 \frac{g(\theta)-g(\theta-\epsilon)}{\epsilon}(|\phi(0)|^2 - |\phi(\theta)||\phi(0)|)\,d\theta
\end{aligned}$$

Note that

$$\lim_{\epsilon\to 0^+}(\int_{-\infty}^0 g_\epsilon(\theta)\phi'(\theta)\,d\theta, \phi(0)) = (A\phi(0), \phi(0)) \tag{2.6}$$ $\boxed{\texttt{sta1}}$

Suppose $|\phi(0)| > |\phi(\theta)|$, then it follows from (2.5)–(2.6) that

$$(A\phi(0), \phi(0)) > 0$$

which is a contradiction to the fact that A is dissipative . That is, $\max|\phi| = |\phi(\theta_0)|$ for some $\theta_0 < 0$ and

$$|\lambda\phi - \mathcal{A}\phi|_Z \geq |\lambda\phi(\theta_0) - \phi'(\theta_0)|_Z \geq \lambda\,|\phi(\theta_0)| = \lambda\,|\phi|_Z$$

since $(\phi'(\theta_0), \phi(\theta_0))_H = 0$. Thus, $\mathcal{A}$ is dissipative. Moreover, for $\lambda > 0$ From (2.3)

$$|I| \leq 2(g(-\delta) + g(-R))|\psi|_Z$$

and thus $\phi = e^{\lambda\theta}\phi(0) + \psi \in dom\,(\mathcal{A})$ satisfies

$$(\lambda\, I - \mathcal{A})\phi = f \quad \text{in } Z,$$

where

$$\begin{aligned}
&\phi(0) = (\Delta(\lambda) - A)^{-1}\int_{-\infty}^0 g(\theta)(f(\theta) - \lambda\int_\theta^0 e^{\lambda(\theta-\xi)}f(\xi)\,d\xi)\,d\theta \\
&\phi = (\lambda\, I - \mathcal{A}^\phi)^{-1}f = e^{\lambda\theta}\phi(0) + \int_\theta^0 e^{\lambda(\theta-\xi)}f(\xi)\,d\xi.\square
\end{aligned} \tag{2.7}$$ $\boxed{\texttt{resl}}$

Hence, we have

thm2.2

Theorem 2.2. *Assume A is m-dissipative. Then, $\mathcal{A}$ is dissipative and $R(\lambda\, I - \mathcal{A}) = Z$ for $\lambda > 0$. Thus, $\mathcal{A}$ generates the C_0-semigroup $T(t)$ on $Z = C((-\infty, 0]; H)$.*

2.3 Banach space-valued solution

In this case section we discuss the case when X is a Banach space and A is m-dissipative linear operator in X. Let X^* is the dual space of X, $\langle\cdot,\cdot\rangle_{X\times X^*}$ denote the dual product and F be the duality mapping

$$F(x)=\{x^*\in X^*:|\langle x,x^*\rangle|=|x|_X|x^*|_{X^*}\}.$$

Then , A is dissipative if and only if for all $x\in dom\,(A)$ there exists a $x^*\in F(x)$, the such that

$$\langle Ax,x^*\rangle\le 0$$

and we assume

$$range(\lambda\,I-A)=X\quad\text{for all }\lambda>0.$$

Then we have the generation theory:

thm2.3 **Theorem 2.3.** *Assume A is m-dissipative in a Banach space X. Then, $\mathcal{A}$ is dissipative and $R(\lambda\,I-\mathcal{A})=Z$ for $\lambda>0$. Thus, $\mathcal{A}$ generates the C_0-semigroup $T(t)$ on $Z=C((-\infty,0];X)$. Also, it follows from (2.7) that*

$$((\lambda\,I-\mathcal{A})^{-1}x)(0)=(\Delta(\lambda)\,I-A)^{-1}\Delta(\lambda)\lambda^{-1}x. \tag{2.8}$$ res2

Proof: First we show that $\mathcal{A}$ is dissipative. For $\phi\in dom\,(\mathcal{A})$ suppose $|\phi(0)|>|\phi(\theta)|$ for all $\theta<0$. For all $x^*\in F(\phi(0))$

$$\begin{aligned}&\langle\int_{-\infty}^0 g_\epsilon(\theta)(\phi')d\theta,x^*\rangle\\ &\quad=\langle\int_{-\infty}^0\frac{g(\theta)-g(\theta-\epsilon)}{\epsilon}\,\langle\phi(\theta)-\phi(0)),x^*\rangle\,d\theta\le 0\end{aligned}$$

since

$$\langle\phi(\theta)-\phi(0),x^*\rangle\le(|\phi(\theta)|-|\phi(0)|)|\phi(0)|<0,\quad\theta<0.$$

Thus,

$$\langle\int_{-\infty}^0 g(\theta)\phi'd\theta,x^*\rangle<0. \tag{2.9}$$ ine0

But, since there exists a $x^*\in F(\phi(0))$ such that

$$\langle Ax,x^*\rangle\ge 0$$

which contradicts to (2.9). Thus, there exists θ_0 such that $|\phi(\theta_0)|=|\phi|_Z$. Since $\langle\phi(\theta),x^*\rangle\le|\phi(\theta)|$ for $x^*\in F(\phi(\theta_0))$, $\theta\to\langle\phi(\theta),x^*\rangle$ attains the maximum at θ_0 and thus $\langle\phi'(\theta_0),x^*\rangle=0$ Hence,

$$|\lambda\,\phi-\phi'|_Z\ge\langle\lambda\,\phi(\theta_0)-\phi'(\theta_0),x^*\rangle=\lambda\,|\phi(\theta_0)|=\lambda\,|\phi|_Z. \tag{2.10}$$ ine1

The range condition is exactly the same as the one in the proof of Theorem (2.2) and the theorem follows from the Lumer-Phillips theorem. □

2.4 Mild solution to (1) frag

Consider the case with the initial value $\phi(\theta) = x \in Z$. It follows from (2.7) resl that for $\mu > 0$

$$(I - \mu\,\mathcal{A})^{-1}x = \phi_\mu, \quad \phi_\mu(\theta) = x + e^{\frac{\theta}{\mu}}(\phi_\mu(0) - x)$$

with

$$\phi_\mu(0) = (\Delta(\frac{1}{\mu})\,I - A)^{-1}\Delta(\frac{1}{\mu})x$$

since

$$\lambda\int_\theta^0 e^{\lambda(\theta-\xi)}\,d\xi - 1 = e^{\lambda\theta}.$$

Note that

$$\phi_\mu(0) - x = (I - \frac{1}{\Delta(\frac{1}{\mu})}\,A)^{-1}\frac{1}{\Delta(\frac{1}{\mu})}Ax$$

Thus,

$$|\phi_\mu(0) - x| \le \frac{1}{\Delta(\frac{1}{\mu})}|Ax|_X \to 0 \quad \text{as } \mu \to 0$$

for all $x \in dom(A)$. Since $\phi_\mu \in dom(\mathcal{A})$, $z_\mu = u_\mu(t+\cdot) = T(t)\phi_\mu$ is a strong solution to

$$\frac{d}{dt}z_\mu(t) = \mathcal{A}z_\mu(t), \quad z_\mu(0) = \phi_\mu.$$

That is, $u_\mu \in C^1((-\infty, T]; X)$ satisfies

$$\int_0^t g(t-s)u_\mu'(s)\,ds = Au_\mu(t) - \int_{-\infty}^0 g(-t+\theta)\phi_\mu'(\theta)\,d\theta.$$

where

$$|\int_{-\infty}^0 g(-t+\theta)\phi_\mu'(\theta)\,d\theta| \le g(t)|\phi_\mu(0) - x| \to 0 \text{ as } \mu \to 0^+.$$

Since D_t^α is closed on $C(0,T;X)$, $u(t) = \lim_{\mu\to 0^+} u_\mu(t)$ satisfies (1.6) fde;

$$D_t^\alpha u = Au(t)$$

Let $g^* = \mathcal{L}^{-1}(\frac{1}{\Delta(\lambda)})$, i.e.,

$$\int_s^\tau g^*(\tau - t)g(t-s)\,dt = 1. \tag{2.11}$$

gs

Thus,

$$\int_0^\tau g^*(\tau-s)\int_0^t g(t-s)u'(s)\,ds = \int_0^\tau u'(s)\int_s^\tau g^*(\tau-t)g(t-s)\,dtds = \int_0^\tau u'\,ds = u(\tau)-u(0). \tag{2.12}$$

gs0

Since $AT(t) = T(t)A$, $Au_\mu(t) \to Au(t)$ in $C(0,T;X)$ and the limit $u(t) = \lim_{\mu\to 0^+} u_\mu(t)$ satisfies

$$u(t) = x + A\int_0^t g^*(t-s)u(s)\,ds. \tag{2.13}$$

int1

Since dom(A) is dense in X and $u(t) \in C(0,T;X)$ defines the mild solution to (1) frag with $f = 0$.

In the case of (1.1), $g^*(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$, i.e,

$$\int_0^\tau \frac{(\tau - t)^{\alpha-1}}{\Gamma(\alpha)} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} = 1 \tag{2.14}$$

J

and we have if $x \in dom(A)$, then $u(t) \in C(0,T; dom(A))$ satisfies

$$u(t) = x + A \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\, ds. \tag{2.15}$$

inte

for (1.1) with $f = 0$.

Theorem 2.4. *Equation (2.13) holds for all $x \in X$, i.e.,*

$$\int_0^t g^*(t-s)u(s)\, ds \in C(0,T; dom\,(A)).$$

In the case of (1.1) if $x \in dom(A)$, then $t \to u(t) = S(t)x \in X$ is absolutely continuous and (2.15) holds for all $x \in X$

Proof: The first assertion follows since $dom\,(A)$ is dense in X and A is a closed operator in X. The second one follows since

$$u'(t) = \frac{d}{dt}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Au(s)\, ds = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Au'(s)\, ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} Ax.\square$$

3 Nonlinear Monotone Equations in Banach spaces

In this section we consider a nonlinear fractional inclusion of the form

$$\int_0^t g(t-s)u'(s)\, ds \in Au(t).$$

Let a graph $A \subset X \times X$ be dissipative, i.e., for any $[x_i, y_i] \in A$ there exists $x^* \in F(x_1 - x_2)$ such that $\mathrm{Re}\langle y_1 - y_2, x^*\rangle \leq 0$, where $F : X \to X^*$ the duality mapping. Or, equivalently

$$|x - \lambda\, y| \geq |x| \quad \text{for all } \lambda > 0 \text{ and } [x,y] \in A.$$

Define $\mathcal{A}$ in $Z = C((-\infty, 0]; X)$ by

$$\mathcal{A}\phi = \phi'$$

with domain

$$dom(\mathcal{A}) = \{\phi' \in Z : \phi(0) \in dom(A),\ \int_{-\infty}^0 g(\theta)\phi'(\theta)\, d\theta \in A\phi(0)\}.$$

thm3.1

Theorem 3.1. *Assume A is dissipative and Range$(\lambda\, I - A)$ for all sufficiently small $\lambda > 0$. Then, $\mathcal{A}$ is dissipative and Range$(\lambda\, I - \mathcal{A}) = Z$ for all sufficiently small $\lambda > 0$. Thus $\mathcal{A}$ generates the nonlinear semigroup of contraction on Z.*

Proof: For $\phi_1,\ \phi_2 \in dom\,(\mathcal{A})$ suppose $|\phi_1(0)-\phi_2(0)| > |\phi_1(\theta)-\phi_2(\theta)|$ for all $\theta < 0$. For all $x^* \in F(\phi_1(0)-\phi_2(0))$

$$\langle \int_{-\infty}^0 g_\epsilon(\theta)(\phi_1'-\phi_2')d\theta, x^*\rangle$$

$$=\langle \int_{-\infty}^0 \frac{g(\theta)-g(\theta-\epsilon)}{\epsilon}\,\langle \phi_1(\theta)-\phi_2(\theta)-(\phi_1(0)-\phi_2(0)),x^*\rangle\,d\theta \le 0$$

since

$$\langle \phi_1(\theta)-\phi_2(\theta)-(\phi_1(0)-\phi_2(0),x^*\rangle \le (|\phi_1(\theta)-\phi_2(\theta)|-|\phi_1(0)-\phi_2(0))|)|\phi_1(0)-\phi_2(0)| < 0, \quad \theta<0.$$

Thus,

$$\langle \int_{-\infty}^0 g(\theta)(\phi_1'-\phi_2')d\theta, x^*\rangle < 0. \tag{3.1}$$

ine

But, since for $y_1 \in A\phi_1(0),\ y_2 \in A\phi_2(0)$ there exists a $x^* \in F(\phi_1(0)-\phi_2(0))$ such that

$$\langle y_1-y_2,x^*\rangle \ge 0$$

which contradicts to (3.1). Thus, there exists θ_0 such that $|\phi_1(\theta_0)-\phi_2(\theta_0)| = |\phi_1-\phi_2|_Z$ and thus $\langle \phi'(\theta_0),x^*\rangle = 0$ for all $x^* \in F(\phi_1(\theta_0)-\phi_2(\theta_0))$. Thus,

$$\begin{aligned} &|\lambda\,(\phi_1-\phi_2)-(\phi'-\phi_2')|_Z \ge \langle \lambda\,(\phi_1(\theta_0)-\phi_2(\theta_0))-(\phi_1'(\theta_0)-\phi_2'(\theta_0)),x^*\rangle \\ &\quad = \lambda\,|\phi_1(\theta_0)-\phi_2(\theta_0)| = \lambda\,|\phi_1-\phi_2|_Z. \end{aligned} \tag{3.2}$$

ineq

For the range condition

$$\lambda\,\phi-\phi' = f, \qquad \int_{-\infty}^0 g(\theta)\phi'(\theta)\,d\theta \in A\phi(0)$$

we have $\phi = e^{\lambda\theta}\phi(0)+\psi$ and

$$\Delta(\lambda)\phi)(0) - \int_{-\infty}^0 g(\theta)\psi'(\theta)\,d\theta \in A\phi(0)$$

where ψ is defined by (2.2). Since A is m-dissipative,

$$\phi(0) = (\Delta(\lambda)\,I-A)^{-1}\int_{-\infty}^0 g(\theta)\psi'(\theta)\,d\theta$$

exits and range$(\lambda\,I-\mathcal{A}) = Z$. Thus, the theorem follows from the Carandall and Ligget theorem [2]. □

3.1 Cone preserving

Let $\mathcal{C}$ be a closed cone in X and A is cone preserving, i.e.,

$$(I-s\,A)^{-1}\mathcal{C} \subset \mathcal{C}$$

for all $s>0$..

thm3.2 **Theorem 3.2.** *$\mathcal{A}$ is cone preserving and $T(t)\mathcal{C} \subset \mathcal{C}$.*

Proof: From

$$\phi = (I - \mu\, A)^{-1} f = e^{\frac{1}{\mu}\theta}\phi(0) + \psi(\theta)$$

with

$$\phi(0) = (\Delta(\frac{1}{\mu}) - A)^{-1}(\int_{-\infty}^{0} g'(\theta)\psi(\theta)\, d\theta)$$

$$\psi(\theta) = \frac{1}{\mu}\int_{\theta}^{0} e^{\frac{1}{\mu}(\theta-\xi)} f(\xi)\, d\xi$$

Thus, since $g' \geq 0$ if $f \in \mathcal{C}$, then $\phi \in \mathcal{C}$. $\square$

4 Nonlinear Fractional Evolution Equations

In this section we consider the case of a class of nonlinear evolution operators $A = A(t)$ for (fee ??). Let X be a Banach space.

$$D_t^g u = \int_0^t g(t-s)u'(s)\, ds \in A(t)u(t), \quad u(0) = x. \tag{4.1}$$ nonle

We assume a family of dissipative operators $A(t) \subset X \times X$, $t \in [0,T]$ satisfy $\overline{dom\,(A(t))} = X$. Define the operator $\mathcal{A}(t)$ in $Z = C((-\infty, 0]; X)$ by

$$\mathcal{A}(t)\phi = \phi'$$

with domain

$$dom\,(\mathcal{A}(t)) = \{\int_{-\infty}^{0} g(\theta)\phi'(\theta)\, d\theta \in A(t)\phi(0)\}.$$

For $\lambda > 0$ define the resolvent

$$J_\lambda(t_i)z = (I - \lambda\, \mathcal{A}(t_i))^{-1}z, \quad z \in Z$$

and $\phi_i = J_\lambda(\mathcal{A}(t_i))z$. Then, $\psi = \phi_1 - \phi_2 \in Z$ satisfies

$$\Delta(\frac{1}{\lambda})\psi(0) = y_1 - y_2$$

for $y_i \in A(t_i)\phi_i(0)$. We assume there exist a continuous function $f : [0,T] \to X$ and a constant $L > 0$ such that

$$|\psi(0)| \leq \lambda^\alpha L\, |f(t_1) - f(t_2)|. \tag{4.2}$$ A.2

Since $\lambda\,\psi - \psi' = 0$, it follows from (ineq 3.2) that

$$|\mathcal{A}_\lambda(t_1)z - \mathcal{A}_\lambda(t_2)z| \leq \lambda^{\alpha-1} L\, |f(t_1) - f(t_2)|, \tag{4.3}$$ cont

where the Yoshida approximation $\mathcal{A}_\lambda(t_1)$ is defined by

$$\mathcal{A}_\lambda(t_i)z = \frac{1}{\lambda}(J_\lambda(t_i)z - z).$$

For $\lambda > 0$ let $\{z_k^\lambda\}$ be the sequence generated by

$$z_k^\lambda = J_\lambda(t_k^\lambda) z_{k-1}^\lambda, \quad z_0^\lambda = \phi \in Z.$$

That is, the product formula $z_k^\lambda = \Pi_{i=1}^m J_\lambda(t_i) z$ defines an approximation sequence and satisfies

$$|\Pi_{i=1}^m J_\lambda(t_i) z_1 - \Pi_{i=1}^m J_\lambda(t_i) z_2| \leq |z_1 - z_2|. \tag{4.4}$$

prod

From (8.4)

$$|\mathcal{A}_\lambda(t_k^\lambda) z_k^\lambda| = |\mathcal{A}_\lambda(t_k^\lambda) J_\lambda(t_k^\lambda) z_{k-1}^\lambda| \leq |\mathcal{A}_\lambda(t_k^\lambda) z_{k-1}^\lambda|$$

$$\leq |\mathcal{A}_\lambda(t_{k-1}^\lambda) z_{k-1}^\lambda| + \lambda^{\alpha-1} L \, |f(t_k^\lambda) - f(t_{k-1}^\lambda)|.$$

If we assume f is of bounded variation on $[0, T]$, then, for

$$a_k = |\mathcal{A}_\lambda(t_k) z_k^\lambda|, \quad b_k = L \, |f(t_k^\lambda) - f(t_{k-1}^\lambda)|$$

we have

$$a_k = a_{k-1} + \lambda^{\alpha-1} b_k$$

and thus

$$|\mathcal{A}_\lambda(t_k^\lambda)| \leq M_0 \lambda^{\alpha-1} \tag{4.5}$$

bound

for all k and λ.

Let $\lambda = 2^{-n}$, $\mu = 2^{-m}$ and $N = 2^{m-n}$ with $t_k^\lambda = k\,\lambda$. For $1 \leq j \leq N$ define $\hat{z}_{iN+j}^\mu$ by

$$\hat{z}_{iN+j}^\mu = J_\mu(t_{(i+1)N}^\lambda) \hat{z}_{iN+j-1}^\mu.$$

It follows from (4.2) that

$$|z_{iN+j}^\mu - \hat{z}_{iN+j}^\mu| \leq |z_{iN+j-1}^\mu - \hat{z}_{iN+j-1}^\mu| + \mu^\alpha |f(t_{(i+1)N}^\mu) - f(t_{iN+j}^\mu)| L.$$

If we assume f is Hölder continuous with order $1 - \alpha + \gamma$, $\gamma > 0$, then

$$\sum_{j=1}^N \mu^\alpha |f(t_{(i+1)N}^\lambda) - f(t_{iN+j}^\lambda)| \leq \lambda 2^{m-n-\gamma\, m}.$$

Define the piecewise constant functions by

$$z_\mu(t) = z_{iN+j}^\mu, \quad \hat{z}_\mu(t) = z_{iN+j}^\mu \quad \text{for } t \in [t_{iN+j}, t_{iN+j+1}).$$

Then,

$$|z_\mu(t) - \hat{z}_\mu(t)| \leq 2^{(1-\gamma)\, m-n} \, L\lambda. \tag{4.6}$$

err1

It follows from Theorem 5.3 (Crandall-Liggett theorem) that

$$|z_\lambda(t) - \hat{z}_\mu(t)| \leq C\,\lambda\, |A_\lambda(t_{i-}^\lambda) z_{i-1}^\lambda|. \tag{4.7}$$

CL

It thus follows from (8.4) that

$$|z_\lambda(t) - \hat{z}_\mu(t)| \leq \tilde{C}\,\lambda^\alpha. \tag{4.8}$$

err2

Hence, from (4.6)–(4.8) we have

$$|z_\lambda(t) - z_\mu(t)| \leq 2^{(1-\gamma)\, m-(2-\alpha)\, n} \, L\lambda^\alpha. + \tilde{C}\lambda^\alpha \leq C\,\lambda^\alpha$$

if $m \le \dfrac{2-\alpha}{1-\gamma} n$. Let us define the sequence $\{m_k\}$ by

$$m_k = [\frac{2-\alpha}{1-\gamma} m_{k-1}], \quad m_0 = n.$$

Then, by induction in k

$$|z_{\mu_k}(t) - z_{\mu_{k-1}}(t)| \le C\, 2^{-\alpha m_{k-1}},$$

where $\mu_k = 2^{-m_k}$. Since $m_k - m_{k-1} \ge \dfrac{1-\alpha+\gamma}{1-\gamma} m_0 > 0$,

$$|z_{\mu_k} - z_{\mu_0}| \le M \lambda^\alpha$$

for some $M > 0$ and thus $\{z_\lambda\}$, $\lambda = 2^{-n}$ is a Cauchy sequence. Consequently, $z(t) = \lim_{\lambda \to 0}$ exits in $C(0,T;Z)$ which defines the solution to (4.1) with $z(0) = \phi \in dom(\mathcal{A}(0))$ and

$$|z_\lambda(t) - z(t)| \le M \lambda^\alpha.$$

It follows from (4.4) that

$$|z_1(t) - z_2(t)| \le |\phi_1 - \phi_2| \quad \text{for } \phi_1,\ \phi_2 \in dom\,(A((0))$$

and $dom\,(A(0))$ is dense in Z, (4.1) has the unique mild solution for all $x \in X$. In summary we have

thm4.1 **Theorem 4.1.** *Assume (A2) and f is of bounded variation and Hölder continuous of order $1-\alpha+\gamma$, $\gamma > 0$ on $[0,T]$. Then,*

$$z(t) = \lim_{\lambda \to 0} \Pi_{i=1}^{[t/\lambda]} J_\lambda(t_i^\lambda) z(0)$$

exits for all $z(0) \in X$. Moreover,

$$|z_\lambda(t) - z(t)| \le C\, \lambda^\alpha$$

for $z(0) \in dom(A(0))$.

The followings are specific cases for which Theorem 4.1. applies. Assume $A(t)$ is of the form

$$A(t)u = Au + g(t,u),$$

where A is a monotone graph and $u \to g(t,u)$ is a montone operator. If

$$|g(t_1,u) - g(t_2,u)|_X \le L\, |f(t_1) - f(t_2)|\, (1 + |u|_X)$$

then, condition (4.2) holds.

Assume for $c > 0$

$$\begin{aligned} &|A(0)A(t)x - A(t)A(0)x| \le c\,|x|. \\ &|A(t_1)x - A(t_2)x| \le |f(t_1) - f(t_2)||A(0)x|. \end{aligned} \tag{4.9}$$ A.3

For $F(t)x = A(0)A(t)x - A(t)A(0)x$ we have

$$D_t^g A(0)x = A(t)A(0)x(t) + F(t)x(t).$$

Thus, if $A(0)x(0) \in X$, one can show that there exists $M > 0$ such that

$$|A(0)x(t)| \le M, \quad t \in [0,T]. \qquad (4.10) \quad \texttt{Bou}$$

Since $\psi(0) = \phi_1(0) - \phi_2(0) \in X$ satisfies

$$\Delta(\lambda)\psi(0) = A(t_2)\psi(0) + (A(t_1) - A(t_2))\phi_1(0),$$

it follows from (A3 ??)–(Bou 4.10) that condition (A2 ??) holds.

5 Operator Theoretic Representation

In this section we develop the solution representation for the linear equation;

$$\int_0^t g(t-s)u'(s) = Au(t) + f(t), \quad u(0) = x. \qquad (5.1) \quad \texttt{Cap}$$

Assume that A is a closed densely defined linear operator A in the Banach space X and there exist $M \ge 1$ and $\omega_0 \in R$ such that

$$|(\lambda\, I - A)^{-1}| \le \frac{M}{Re\,\lambda - \omega_0} \quad \text{for all } Re\,\lambda\omega_0. \qquad (5.2) \quad \texttt{res}$$

For example, (res 5.2) holds if A generates a C_0 semigroup of $G(M, \omega_0)$ type on X. Define the Yoshida approximation $A_\mu \in \mathcal{L}(X)$ of A by

$$A_\mu = A(I - \mu\, A)^{-1} \quad \text{for } \mu\omega_0 < 1. \qquad (5.3) \quad \texttt{yoshida}$$

Then, $A_\mu \in \mathcal{L}(X)$ and consider the equation

$$\int_0^t g(t-s)u_\mu'(s)\,ds = A_\mu u_\mu(t) + f(t), \quad u_\mu(0) = x.$$

Taking the Laplace transform of the equation, we obtain

$$\Delta(\lambda)\hat{u}_\mu - \lambda^{-1}\Delta(\lambda)x = A_\mu \hat{u}_\mu + \hat{f},$$

and thus

$$\hat{u}_\mu = (\Delta(\lambda)\, I - A_\mu)^{-1}\lambda^{-1}\Delta(\lambda)x + (\Delta(\lambda)\, I - A_\mu)^{-1}\hat{f}.$$

Let $f = 0$. Since $\mathrm{Re}\,\sigma(A_\mu) \le \dfrac{\omega_0}{1 - \mu\omega_0} < \gamma$, we have

$$u_\mu(t) = S_\mu(t)x = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}(\Delta(\lambda) - A_\mu)^{-1}\lambda^{-1}\Delta(\lambda)x\,d\lambda.$$

Note that

$$\Delta(\lambda)(\Delta(\lambda)\, I - A)^{-1} = I + (\Delta(\lambda)\, I - A)^{-1}A \qquad (5.4) \quad \texttt{id}$$

and in general

$$\Delta(\lambda)(\Delta(\lambda)\, I - A)^{-1} = \sum_{k=0}^{n-1} \Delta(\lambda)^{-k} A^k + (\Delta(\lambda)\, I - A)^{-1} \Delta(\lambda)^{-n} A^n. \qquad (5.5)$$ ser

Since

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t}}{\lambda}\, d\lambda = 1$$

and

$$\int_{\gamma - i\infty}^{\gamma + i\infty} |\lambda^{-1} \Delta(\lambda)^{-1}|\, d\lambda < \infty,$$

we have

$$|S_\mu(t) x| \le M\, |Ax|,$$

uniformly in $\mu > 0$. Since from (id 5.4)

$$\Delta(\lambda)\lambda^{-1}(\Delta(\lambda)\, I - A)^{-1} = \frac{1}{\lambda} + \lambda^{-1} A (\Delta(\lambda)\, I - A)^{-1},$$

if $\mathcal{L}(\Delta(\lambda)^{-1}) = g^*$ we have

$$u_\mu(t) = x + \int_0^t g^*(t-s)\, A_\mu S_\mu(s) x\, ds.$$

Moreover, since

$$(\Delta(\lambda)\, I - A_\mu)^{-1} x - (\Delta(\lambda)\, I - A)^{-1} x = \frac{\mu}{1 + \mu \Delta(\lambda)} (\nu\, I - A)^{-1} (\Delta(\lambda)\, I - A)^{-1} A^2 x,$$

where $\nu = \dfrac{\Delta(\lambda)}{1 + \mu\Delta(\lambda)}$, $\{u_\mu(t)\}$ is Cauchy in $C(0,T;X)$ provided that $x \in dom(A^3)$. Letting $\mu \to 0^+$, we obtain

thm5.1 **Theorem 5.1.** *For $x \in dom(A^2)$*

$$u(t) = S(t)x = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (\Delta(\lambda)\, I - A)^{-1} \lambda^{-1} \Delta(\lambda) x\, d\lambda$$

$$u(t) = x + \int_0^t g^*(t-s) A S(s) x\, ds. \qquad (5.6)$$ res1

corr5.1 **Corollary 5.1.** *For $x \in dom(A^{n+2})$,*

$$u(t) = S(t)x = x + \sum_{k=1}^{n-1} \mathcal{L}^{-1}(\lambda^{-1}\Delta(\lambda)^{-k})\, A^k x + \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (\Delta(\lambda)\, I - A)^{-1} \lambda^{-1} \Delta(\lambda)^{-n} A^n x\, d\lambda, \qquad (5.7)$$ res2

x where $\mathcal{L}^{-1}(\cdot)$ is the inverse Laplace transform. In the case of the Caputo ($g = g_{1-\alpha} = \dfrac{|\theta|^{-\alpha}}{\Gamma(1-\alpha)}$)

$$S(t)x = \sum_{k=0}^{n-1} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}\, A^k x + \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (\lambda^\alpha\, I - A)^{-1} \lambda^{-1-n\alpha}\, A^n x\, d\lambda,$$

and

$$u(t) = x + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AS(t)x\,dx.$$

Proof: The first assertion follows from (5.5). For the Caputo case $\Delta(\lambda) = \lambda^\alpha$ and $g^*(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$. $\square$

5.1 Riemann-Liouville equation

In this section we consider the Riemann-Liouville equation

$$\frac{d}{dt}\int_0^t g(t-s)u(s)\,ds = Au(t) + f(t) \tag{5.8}$$ SL

with

$$(\int_0^t g(t-s)u(s)\,ds)(0^+) = y.$$

Taking the Laplace transform, we have

$$\Delta(\lambda)\hat{u} = y + A\hat{u} + \hat{f}$$

and thus

$$\hat{u} = (\Delta(\lambda)\,I - A)^{-1}(y + \hat{f}) \tag{5.9}$$ Lap

Note that

$$\Delta(\lambda)(\Delta(\lambda)\,I - A)^{-1} = I + (\Delta(\lambda)\,I - A)^{-1}A$$

Let the linear operator $P(t)$

$$P(t)y = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\Delta(\lambda)\,I - A)^{-1}y\,d\lambda,$$

for $y \in dom(A)$. Hence, using the same arguments as above, from (5.9) we have;

th5.2 **Theorem 5.2.** *The solution to (5.8) is given by*

$$u(t) = P(t)y + \int_0^t P(t-s)f(s)\,ds$$

for $y \in dom(A)$ and $f \in L^2(0,T;dom(A)$, and for the case of fractional derivative $g = g_{1-\alpha} = \dfrac{|\theta|^{-\alpha}}{\Gamma(1-\alpha)}$;

$$P(t)y = t^{\alpha-1}\sum_{k=0}^{n-1}\frac{t^{k\alpha}}{\Gamma(k\alpha+\alpha)}\,A^k y + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(\lambda^\alpha\,I - A)^{-1}\lambda^{-n\alpha}\,A^n y\,d\lambda.$$

for $y \in dom(A^{n+2})$.

Proof: The second assertion follows from

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\lambda^{-\alpha}\,d\lambda=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

and the fact that

$$(\lambda^\alpha I-A)^{-1}x=\lambda^{-\alpha}x+(\lambda^\alpha I-A)^{-1}\lambda^{-\alpha}$$

and by induction

$$(\lambda^\alpha I-A)^{-1}x=\lambda^{-\alpha}\sum_{k=0}^{n-1}\left(\frac{A}{\lambda^\alpha}\right)^k+(\lambda^\alpha I-A)^{-1}\left(\frac{A}{\lambda^\alpha}\right)^n x.\square$$

Similarly, for (Cap 5.1) the solution is given by

$$u(t)=S(t)x+\int_0^t P(t-s)f(s)\,ds \tag{5.10}$$ Sol

for $x\in dom(A^2)$ and $f\in C(),T;X)$

cor5.2 **Corollary 5.2.**

$$\frac{d}{dt}S(t)=P(t)A$$

$$S(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}P(s)\,ds.$$

Proof: The first assertion follows from the fact that

$$\lambda^\alpha(\lambda^\alpha I-A)^{-1}=I+(\lambda^\alpha I-A)^{-1}A.$$

The second one follows simply follows from

$$S(t)x=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha I-A)^{-1}x\,d\lambda$$

and

$$\mathcal{L}^{-1}(\lambda^{\alpha-1})=\frac{t^{\alpha-1}}{\Gamma(\alpha)}.\square$$

5.2 Sectorial Calculus and Asymptotic Estimates

Let a closed and densely defined linear operator A satisfy

$$|(\lambda I-A)^{-1}|\le\frac{M}{Re\,\lambda}\quad\text{for all } Re\,\lambda>0 \tag{5.11}$$ Res-e

Assume there exits $\alpha\in(0,1)$ such that for $0<c_1\le c_2<\infty$

$$c_1\,|\lambda^\alpha|\le|\Delta(\lambda)|\le c_2\,|\lambda^\alpha|,\quad \mathrm{Re}\lambda>\omega_0. \tag{5.12}$$ ass

We recall $\Delta(\lambda)=\lambda^\alpha$ for

$$g=g_{1-\alpha}=\frac{|t|^{-\alpha}}{\Gamma(1-\alpha)}.$$

Then, for $0 < \alpha < 1$ there exists $\theta_0 > 0$ such that

$$|(\Delta(\lambda)\, I - A)^{-1}| \leq \frac{M_\alpha}{|\lambda|^\alpha} \quad \text{on } \Sigma_\theta = \{z \in C : \arg(z) < \frac{\pi}{2} + \theta_0\} \cap \{z \neq 0\}$$

since

$$Re\, \lambda^\alpha \geq |\lambda|^\alpha \cos(\alpha\theta) \quad \text{for } \lambda = |\lambda| e^{i\theta}.$$

Let $\Gamma_{\gamma,\delta}$ be the integration path defined by

$$\Gamma^\pm = \{z \in C : |z| \geq \delta,\ arg(z) = \pm(\frac{\pi}{2} + \theta)\}, \quad \Gamma_0 = \{z \in C : |z| = \delta,\ |arg(z)| \leq \frac{\pi}{2} + \theta\}$$

for some $\delta > 0$ and $0 < \theta \leq \theta_0$. Then, the solution map $S(t) : x \in X \to u(t) \in X$ is given by

$$S(t)x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\Delta(\lambda)\, I - A)^{-1} \Delta(\lambda) \lambda^{-1} x \, d\lambda, \tag{5.13}$$ Rep1

using the Cauchy integral representation ([res1] 5.6) and the analytic continuation. On $\Gamma^\pm$

$$|e^{\lambda t} (\Delta(\lambda)\, I - A)^{-1} \Delta(\lambda) \lambda^{-1}| \leq \frac{M}{r} e^{-r \sin\theta\, t}$$

for $\lambda = r(cos(\theta) + i \sin(\theta)),\ r \geq \delta$. On Γ_0

$$|e^{i\lambda t} (\Delta(\lambda)\, I - A)^{-1} \Delta(\lambda) \lambda^{-1}| \leq \frac{M}{\delta} e^{\delta \sin\phi\, t}$$

for $\lambda = \delta(cos(\phi) + i \sin(\phi)),\ |\phi| \leq \frac{\pi}{2} + \theta$. Hence ([Rep1] 5.13) holds for all $x \in X$. Let $P(t) \in \mathcal{L}(X),\ t > 0$ be

$$P(t)x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\Delta(\lambda)\, I - A)^{-1} x \, d\lambda.$$

Since

$$\Delta(\lambda)\, (\Delta(\lambda)\, I - A)^{-1} = I + A(\Delta(\lambda)\, I - A)^{-1},$$

if $x \in dom(A)$, then $u(t) = S(t)x \in C(0,T;X)$ for $x \in dom(A)$ satisfies

$$\frac{d}{dt} S(t)x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\Delta(\lambda)\, I - A)^{-1} \Delta(\lambda) x \, d\lambda = P(t)Ax.$$

and if $g^* = \mathcal{L}^{-1}(\Delta(\Lambda)^{-1})$, then

$$g^*(t) P(t) x \in C(0,T;X).$$

For $x \in X$ and $f \in C(0,T;X)$ we have ([Sol] 5.10):

$$u(t) = S(t)x + \int_0^t P(t-s) f(s) \, ds$$

for the solution to ([Cap] 5.1).

thm5.3

Theorem 5.3. *Assume $0 \in \rho(A)$. Then, for some $C > 0$*

$$|\frac{d}{dt}S(t)x| \leq \frac{C}{t}\,|x|,$$

$$|P(t)x| \leq \frac{C}{t^{1-\alpha}}\,|x|,$$

$$AS(t)x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}((\Delta(\lambda)\,I - A)^{-1}\Delta(\lambda)^2\frac{1}{\lambda} - \Delta(\lambda)\frac{1}{\lambda})x\,d\lambda.$$

Proof: Since $0 \in \rho(A)$, one can let $\delta = 0$ for the integral path Γ and

$$|\Delta(\lambda)(\Delta(\lambda)\,I - A)^{-1}| \leq M \text{ on } \Gamma.$$

Thus,

$$|\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\Delta(\lambda)\,I - A)^{-1}\Delta(\lambda)x\,d\lambda| \leq \frac{M}{\pi}\int_0^\infty e^{-r\sin\theta\,t}\,dr \leq \frac{M}{\pi\sin\theta\,t}.$$

$$|P(t)| = |\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\Delta(\lambda)\,I - A)^{-1}\,d\lambda| \leq \frac{M}{\pi}\int_0^\infty r^{-\alpha}e^{-r\sin\theta\,t}\,dr \leq \frac{M}{\pi}\Gamma(1-\alpha)(\sin\theta\,t)^{\alpha-1}.\square$$

5.3 Caputo equation and Inverse inequality

Consider the case when

$$g(t-s) = g_{1-\alpha}(t-s) = \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}.$$

thm5.4

Theorem 5.4. *For $t > 0$ $R(S(t)) \subset R(A^{-1})$. If $R(A^{-1})$ is pre-compact and $S(t)$ is injective, then $R(S(t)) = R(A^{-1})$.*

Proof: Since

$$A(\lambda^\alpha\,I - A)^{-1} = \lambda^\alpha(\lambda^\alpha\,I - A)^{-1} - I, \tag{5.14}$$ Eq

we have

$$AS(t)x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{2\alpha-1}(\lambda^\alpha\,I - A)^{-1}x\,d\lambda - \frac{1}{\Gamma(1-\alpha)}t^{-\alpha}\,x = Kx - g_{1-\alpha}(t)x$$

where

$$Kx = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{2\alpha-1}(\lambda^\alpha\,I - A)^{-1}x\,d\lambda$$

Then, $R(S(t)) \subset R(A^{-1})$. Since

$$AK = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{3\alpha-1}(\lambda^\alpha\,I - A)^{-1}x\,d\lambda - \frac{t^{-2\alpha}}{\Gamma(1-2\alpha)} \in \mathcal{L}(X),$$

if $R(A^{-1})$ is pre-compact, K is a compact operator. Moreover, if $S(t)$ is injective, then it follows from the Fredholm alternative theorem that $R(S(t)) = R(A^{-1})$. $\square$

cor5.3 **Corollary 5.3.** *For $t > 0$*

$$S(t) = -\sum_{k=1}^{n-1} \frac{t^{-k\alpha}}{\Gamma(1-k\alpha)} A^{-k} + \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{n\alpha-1}(\lambda^\alpha I - A)^{-1}A^{-n}\,d\lambda. \tag{5.15}$$ ser1

For $t > 0$ sufficiently large $AS(t) = K - g_{1-\alpha}(t)\,I \in \mathcal{L}(X)$ is bounded invertible, i.e, there exits a constant c such that

$$|x| \le c\,|AS(t)x|.$$

Moreover, for $x \in X$

$$|S(t)x| \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,|A^{-1}x|.$$

Proof: Equation (sere ??) follows from

$$A^n(\lambda^\alpha I - A)^{-1} = \lambda^{n\alpha}(\lambda^\alpha I - A)^{-1} - \sum_{k=0}^{n-1}\lambda^{k\alpha}A^k.$$

Thus, $g_{1-\alpha}(t) - |K| > 0$ for sufficiently large $t > 0$. and $|(g_{1-\alpha}(t)\,I - K)^{-1}| \le (g_{1-\alpha}(t) - |K|)^{-1}$. □

Similarly, we have

cor5.4 **Corollary 5.4.** *For $t > 0$*

$$|AS(t)| \le \frac{M}{\Gamma(1-\alpha)}t^{-\alpha}$$

$$|P(t)| \le \frac{M}{\Gamma(\alpha)}t^{-(1-\alpha)}.$$

$$|AP(t)| \le Mt^{-1}$$

Moreover, assume A is a sectorial operator, i.e., there exist $M > 0$, $\theta_0 > 0$ such that

$$|(z\,I - A)^{-1}| \le \frac{M}{|z|} \quad \text{on } \Sigma_{\theta_0} = \{z \in C : arg(z) \le \frac{\pi}{2} + \theta_0\} \cap \{z \ne 0\}$$

Then,

$$A^\beta x = \int_\Gamma \lambda^\beta (z\,I - A)^{-1}x$$

and for $0 \le \beta \le 1$

$$|A^\beta P(t)| \le \frac{M}{t^{1-\alpha+\beta\alpha}}. \tag{5.16}$$ sec-frac

Proof: It follows from (Eq 5.14). The last assertion uses $|A^\beta x| \le M\,|Ax|^\beta |x|^{1-\beta}$ for $x \in dom(A)$. □

5.4 Basset equation

Consider the Basset equation

$$u'(t) + kD_t^\alpha u = Au(t) + f(t), \quad u(0) = x.$$

In this case

$$\Delta(\lambda) = \lambda + k\,\lambda^\alpha.$$

Assume A a sectorial operator so that (5.13) holds. Since

$$A(\Delta(\lambda) - A)^{-1}\lambda^{-1}\Delta(\lambda) = (\Delta(\lambda) - A)^{-1}\lambda^{-1}\Delta(\lambda)^2 - (1 + k\lambda^{\alpha-1})\,I$$

we have

$$AS(t)x = Kx - k\,g_{1-\alpha}(t)x,$$

where

$$Kx = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{-1}\Delta(\lambda)^2(\Delta(\lambda)\,I - A)^{-1}x\,d\lambda.$$

Moreover, we have the estimate:

$$|K| \leq M((t\sin\theta)^{-1} + k\Gamma(\alpha)(t\sin\theta)^{-\alpha}).$$

Thus, if A^{-1} is compact, Theorem 5.4 holds.

6 Series Solution

Define the operator:

$$J_t^\alpha\phi = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)\,ds$$

Since $J_t^{1-\alpha}J_t^\alpha = J_t^1$, we have

$$D_t^\alpha J_t^\alpha\phi = \phi \quad \text{and} \quad J_t^\alpha D_t^\alpha\phi = \phi - \phi(0).$$

Thus, (1.1) is equivalently written as

$$u(t) = x + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}Au(s)\,ds \tag{6.1}$$

Thus,

$$S(t)x = x + A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}S(s)x\,ds. \tag{6.2}$$

Since $Au \in C(0,T;H)$ for $x \in dom(A)$ it follows from (6.1) that

$$\lim_{t\to 0^+}\frac{u(t)-x}{t^\alpha} = \frac{1}{\Gamma(\alpha+1)}Ax$$

one has the first order approximation

$$u(t) \sim (I - \frac{t^\alpha}{\Gamma(\alpha+1)}\,A)^{-1}x.$$

Moreover, we have

$$\lim_{t\to 0^+}\frac{u(t)-x-\frac{1}{\Gamma(\alpha+1)}Ax}{t^{2\alpha}}=\frac{1}{\Gamma(2\alpha+1)}A^2x$$

we look for an second order approximation of the form

$$u(t)\sim (I+b_1t^\alpha\,A)^{-1}(I+c_1t^\alpha\,A)x$$

with

$$c_1-b_1=\frac{1}{\Gamma(\alpha+1)},\quad b_1^2-c_1b_1=\frac{1}{\Gamma(2\alpha+1)},$$

Thus, we obtain

$$b_1=-\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)},\quad c_1=\frac{1}{\Gamma(\alpha+1)}-\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}.$$

In general we can find the Pade(n,n) approximation of the form

$$u(t)\sim (\sum_{k=0}^n b_kt^{k\alpha}\,A^k)^{-1}(I+\sum_{k=1}^n a_kt^{k\alpha}\,A^k)x.$$

That is, since

$$E_{\alpha,1}(\lambda t)=\sum_{j=0}^\infty c_j\lambda^kt^{j\alpha},\quad c_j=\frac{1}{\Gamma(j\alpha+1)}.$$

we have

$$(\sum_{k=0}^n b_kt^{k\alpha})(\sum_{j=0} c_jt^{j\alpha})=\sum_{\ell=0}^n a_\ell t^{\ell\alpha}$$

in term by term, i.e.,

$$a_\ell=\sum_{k=0}^\ell b_kc_{\ell-k},\quad 0\le\ell\le n+n.$$

Thus, $\{b_k\}$ satisfies

$$\begin{pmatrix} c_n & c_{n-1} & \dots & c_0\\ c_{n+1} & c_{n-1} & \dots & c_1\\ & \vdots & & \\ c_{n+n} & c_{n-1} & \dots & c_n\end{pmatrix}\begin{pmatrix} b_0\\ b_1\\ \vdots\\ b_n\end{pmatrix}=\begin{pmatrix} a_n\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

Next, we consider the series expansion in terms of the resolvent $(\lambda^\alpha\,I-A)^{-1}$. The Post-Widder inversion theory is given as:

FW

Theorem 6.1. *Let $u(t)$ be a X-valued continuous function on $t\ge 0$ such that $u(t)=O(e^{\gamma t})$ as $t\to\infty$ for some γ and $\hat{u}$ be the Laplace transform of $u(t)$. Then,*

$$u(t)=\lim_{n\to\infty\infty}\frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1}(\frac{\partial^n}{\partial\lambda^n}\hat{u})(\frac{n}{t}),$$

uniformly on any compact sets of $t>0$.

thm5.6

Theorem 6.2. *If A generates a C_0 semigroup on X, then*

$$\frac{\partial^n}{\partial\lambda^n}(\lambda^{\alpha-1}(\lambda^\alpha\, I-A)^{-1})=(-1)^{-(n+1)}\lambda\!-\!(n+1)\sum_{k=1}^{n+1}b^\alpha_{k,n+1}(\lambda^\alpha(\lambda^\alpha\, I-A)^{-1})^k$$

where $b^\alpha_{k,n}$ are given by the recurrence

$$b^\alpha_{k,n}=(n-1-k\alpha)b^\alpha_{k,n-1}+\alpha(k-1)b^\alpha_{k-1,n-1},\ \ 1\le k\le n,\ \ n\ge 2$$

with $b^\alpha_{1,1}=1$ and $b^\alpha_{k,n}=0,\ k>n$, and

$$S(t)x=\lim_{n\to\infty}\frac{1}{n!}\sum_{k=1}^{n+1}b^\alpha_{k,n+1}\,(I-(\frac{t}{n})^\alpha\, A)^{-k}x$$

where the convergence is uniform on bounded intervals of $t\ge 0$.

Proof: By induction in n we have

$$\frac{\partial^n}{\partial\lambda^n}(\lambda^{\alpha-1}(\lambda^\alpha\, I-A)^{-1})=(-1)^n\lambda^{-(n+1)}\sum_{k=1}^{n+1}b^\alpha_{k,n+1}(\lambda^\alpha(\lambda^\alpha\, I-A)^{-1})^k$$

Since $b^\alpha_{k,n}>0$ for $\alpha\in(0,1)$, we have

$$|\frac{\partial^n}{\partial\lambda^n}(\lambda^{\alpha-1}(\lambda^\alpha\, I-A)^{-1})|\le\sum_{k=1}^{n+1}b^\alpha_{k,n+1}\lambda^{k\alpha-n-1}|(\lambda^\alpha\, I-A)^{-k}|$$

$$\le M\sum_{k=1}^{n+1}b^\alpha_{k,n+1}\frac{\lambda^{k\alpha-n-1}}{(\lambda^\alpha-w)^k}=M(-1)^n\frac{\partial^n}{\partial\lambda^n}(\frac{\lambda^{\alpha-1}}{\lambda^\alpha-\omega})$$

Since

$$\frac{\lambda^{\alpha-1}}{\lambda^\alpha-\omega}=\int_0^\infty e^{-\lambda t}E_{\alpha,1}(t)\,dt,$$

$$(-1)^n\frac{\partial^n}{\partial\lambda^n}(\frac{\lambda^{\alpha-1}}{\lambda^\alpha-\omega})=\int_0^\infty t^ne^{-\lambda t}E_{\alpha,1}(t)\,dt\le C\,\frac{n!}{(\lambda-\omega^{\frac{1}{\alpha}})^{n+1}}$$

since $E_{\alpha,1}(\omega t^\alpha)\le Ce^{\frac{1}{\alpha}t}$. Thus,

$$|\frac{\partial^n}{\partial\lambda^n}(\lambda^{\alpha-1}(\lambda^\alpha\, I-A)^{-1})|\le CM\,\frac{n!}{(\lambda-\omega^{\frac{1}{\alpha}})^{n+1}}$$

Since $u(t)=S(t)x$ is continuous function on $t\ge 0$ and $u(t)=O(e^{\gamma t}$ as $t\to\infty$, the theorem follows from the Post-Widder inversion theory. □

Similarly, we have

cor5.5

Corollary 6.1.

$$P(t)=\frac{d}{dt}S(t)A^{-1}=\lim_{n\to\infty}\frac{t^{\alpha-1}}{n!}\sum_{k=1}^{n+1}\frac{k}{n^\alpha}b^\alpha_{k,n+1}\,(I-(\frac{t}{n})^\alpha\, A)^{-(k+1)}x.$$

*where the convergence is uniform on bounded intervals of $t > 0$. The solution to (**??**) is given by*

$$u(t) = S(t)x + \int_0^t P(t-s)f(s)\,ds.$$

corr5.6 **Corollary 6.2.** *For For the Caputo equation with $A \in \mathcal{L}(X)$*

$$S(t) = E_{\alpha,1}(At^\alpha) = \sum_{n=0}^\infty \frac{A^n t^{n\alpha}}{\Gamma(n\alpha+1)}$$

and

$$P(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = t^{\alpha-1}\sum_{n=0}^\infty \frac{A^n t^{n\alpha}}{\Gamma(n\alpha+\alpha)}.$$

7 Nonhomogeneous equation

Consider the nonhomogeneous equation

$$\int_0^t g(t-s)u'(s)\,ds = Au(t) + f(t), \quad u(0) = 0 \tag{7.1}$$ nhom

Suppose $u \in L^2_{loc}(0,\infty,X)$ be a solution. Then, it is unique and satisfies the Laplace transform of (7.1)

$$(\Delta(\lambda)\,I - A)\hat{u} = \hat{f}. \tag{7.2}$$ lap

It follows from Theorem 2.3 that $\mathcal{A}$ generates the C_0 semigroup $T(t),\ t \geq 0$ on $Z = C((-\infty]; X)$.

thm **Theorem 7.1.** *For $\omega \geq 0$ we define the operator*

$$\Psi_\omega = e^{\omega s}(A - \Delta(\omega)\,I)^{-1} \in \mathcal{L}(X,Z)$$

and define

$$u(t) = \left((\mathcal{A} - \omega\,I)\int_0^t T(t-s)\Psi_\omega f(s)\,ds\right)(0). \tag{7.3}$$ nhom1

Then, (7.3) defines the solution to (7.1) if $f \in C^2(0,T;X)$.

Proof: Note that

$$z(t) = (\mathcal{A} - \omega\,I)\int_0^t T(t-s)\Psi_\omega f(s)\,ds$$

$$= T(t)\Psi_\omega f(0) - \Psi_\omega f(t) + \int_0^t T(t-s)\Psi_\omega f'(s)\,ds - \omega\int_0^t T(t-s)\Psi_\omega f(s)\,ds \tag{7.4}$$ lap1

and thus, $z \in C(0,T;Z)$ and $z(t) = u(t+\cdot)$. Taking the Laplace transform of (7.4), we obtain

$$\hat{z}(\lambda) = (\mathcal{A} - \omega\,I)(\lambda\,I - \mathcal{A})^{-1}\Psi_\omega \hat{f}(\lambda)$$

$$= (\lambda - \omega)(\lambda\,I - \mathcal{A})^{-1}\Psi_\omega \hat{f}(\lambda) - \Psi_\omega \hat{f}(\lambda).$$

Since from (5.6) res1 and

$$\Delta(\lambda)=\lambda\int_{-\infty}^0 e^{\lambda\theta}g(\theta)\,d\theta,$$

$$((\lambda\,I-\mathcal{A})^{-1}\Psi_\omega)(0)=-(\Delta(\lambda)\,I-A)^{-1}(\Delta(\omega)\,I-A)^{-1}\int_{-\infty}^0 g(\theta)(e^{\omega\theta}-\lambda\int_\theta^0 e^{\lambda(\theta-\xi)}e^{\omega\xi}\,d\xi)$$

$$=(\Delta(\lambda)\,I-A)^{-1}(\Delta(\omega)\,I-A)^{-1}\frac{\Delta(\omega)-\Delta(\lambda)}{\lambda-\omega},$$

Thus we have

$$\hat{u}(\lambda)=(\Delta(\lambda)\,I-A)^{-1}\hat{f}(\lambda)$$

and the claim holds. □

When $0\in\rho(A)$, we let $\omega=0$ and obtain

$$\begin{aligned} u(t)&=S(t)A^{-1}f(0)-A^{-1}f(t)+\int_0^t S(t-s)A^{-1}f'(s)\,ds \\ &=\int_0^t \dot{S}(t-s)A^{-1}f(s)\,ds=\int_0^t P(t-s)f(s)\,ds, \end{aligned} \tag{7.5}$$ nhom0

for $f\in W^{11}(0,T;X)$, since

$$\dot{S}=\frac{d}{dt}S(t)=P(t)A.$$

Otherwise, for $\omega\geq 0$ let $y(t)=e^{-\omega t}u(t)$. It can be easily seen that y satisfies

$$\int_{-\infty}^t g(t-s)e^{-\omega(t-s)}y'(s)\,ds=-\omega\int_{-\infty}^t g(t-s)e^{-\omega(t-s)}y(s)\,ds+Ay(t)+e^{-\omega t}f(t)$$

with $y(s)=0,\ s\leq 0$. Since

$$\int_0^t g_\epsilon(t-s)e^{-\omega(t-s)}y'(s)\,ds=\int_{-\infty}^t (g_\epsilon'(t-s)-\omega\,g_\epsilon(t-s))e^{-\omega(t-s)}(y(s)-y(t))\,dt,$$

we have

$$(\Delta(\omega)\,I-A)y(t)-\lim_{\epsilon\to 0^+}\int_{-\infty}^t g_\epsilon'(t-s)e^{-\omega(t-s)}(y(t)-y(s))\,ds=e^{-\omega t}f(t)$$

Suppose $|y(t)|=\max_{s\leq T}|y(s)|$ we have

$$\langle(\Delta(\omega)\,I-A)y(t),x^*\rangle\leq e^{-\omega t}\langle f(t),x^*\rangle$$

for $x^*\in F(y(t))$ since $g_\epsilon'\leq 0$ on R^+ and

$$\langle y(t)-y(s),x^*\rangle\geq(|y(t)|-|y(s)|)|y(t)|\geq 0.$$

Since A is dissipative we have

$$|y(t)|\leq|\Delta(\omega)^{-1}|\,e^{-\omega t}|f(t)|. \tag{7.6}$$ est

thm6.2

Theorem 7.2. *For $f \in C(0,T;X)$ there exists a unique solution u in $C(0,T;X)$ to (7.1) and it has the representation (7.5);*

$$u(t) = \int_0^t P(t-s)f(s)\,ds.$$

and for some $M >$

$$|u(t)| \leq M\,|f|_{C(0,t;X)}.$$

Proof: Since $C^2(0,T;X)$ is dense in $C(0,T;X)$ and $\mathcal{A}$ is closed, the theorem follows from (7.6).

7.1 $f \in L^2(0,T;H)$

Let $X = H$ be a Hilbert space. Since for $R > T$ and $0 \leq t \leq T$ and $C^1(0,T;H)$,

$$\int_{t-R}^t g(t-s)(u'(s), u(s) - u(t))\,ds = \int_{-R}^0 g(-\theta)(u'(t+\theta), u(t+\theta) - u(t))\,d\theta$$

$$= \frac{1}{2}(\int_{-R}^0 g'(-\theta)|u(t+\theta) - u(t)|^2\,d\theta - g(R)|u(t) - u(0)|^2)$$

and

$$\int_{t-R}^t g(t-s)(u'(s), u(s))\,ds = \int_{-R}^0 g(-\theta)(u'(t+\theta), u(t+\theta))\,d\theta$$

$$= \frac{d}{dt}\frac{1}{2}\int_{-R}^0 g(-\theta)|u(t+\theta)|^2\,d\theta,$$

it follows from

$$\int_)^T (\int_0^t g(t-s)u'(s)\,ds, u(t))\,dt \int_0^T \int_0^t g(t-s)(u'(s), u(t)-u(s))\,dsdt + \int_0^T \int_o^t g(t-s)u'(s), u(s))\,dsdt$$

that

$$\int_{-R}^0 g(\theta)|u(T+\theta)|^2\,d\theta - \int_{-R}^0 g(\theta)|x|^2\,d\theta$$

$$+ \int_0^T (\int_{-R}^0 g'(\theta)|u(t+\theta) - u(t)|^2\,d\theta + g(-R)|x - u(t)|^2)\,dt \tag{7.7}$$

enrg

$$= 2\int_0^T \int_0^t g(t-s)u'(s)\,ds, u(t))\,dt.$$

Letting $R \to T^+$, we obtain

$$\int_{-T}^0 g(\theta)|u(T+\theta)|^2\,d\theta - \int_{-T}^0 g(\theta)|x|^2\,d\theta$$

$$+ \int_0^T (\int_{-t}^0 g'(\theta)|u(t+\theta) - u(t)|^2\,d\theta + g(T)|x - u(t)|^2)\,dt = 2\int_0^T \int_0^t g(t-s)u'(s)\,ds, u(t))\,dt. \tag{7.8}$$

enrg1

Thus, we have

cor6.1 **Theorem 7.3.** *The energy identity (7.8) holds for all $u \in L^2(0,T;H)$ satisfying $\int_0^t g(t-s)u'(s)\,ds \in L^2(0,T;H)$ and for all $u(0)=0$.*

$$\int_0^T (\int_0^t g(t-s)u'(s)\,ds, u(t))\,dt \geq g(T)\,|\int_0^T |u(t)|^2\,dt).$$

Assume that there exists $\omega > 0$ such that

$$(A\phi,\phi) \leq -\omega\,|\phi|^2, \quad \text{for all } \phi \in dom\,(A). \tag{7.9}$$ ass

Suppose $f(0)=0$ and $f \in C^2(0,T;H)$ it follows from (7.5) that $u \in C^1(0,T;H) \cap C(0,T;dom(A))$. Then, from (??)

$$\int_{-T}^0 g(\theta)|u(T+\theta)|^2\,d\theta - \int_{-t}^0 g(\theta)|x|^2\,d\theta$$

$$+\int_0^T (-\int_{-T}^0 g'(\theta)|u(t+\theta)-u(t)|^2\,d\theta + g(T)\,|x-u(t)|^2 + \omega\,|u(t)|^2)\,dt \leq \frac{1}{\omega}\int_0^T |f(t)|^2\,dt.$$

Otherwise, using exactly the same arguments as for the estimate for $C(0,T;H)$, we have

$$\int_{-\infty}^0 g(\theta)e^{\omega\theta}|y(T+\theta)|^2\,d\theta - \frac{\Delta(\omega)}{\omega}|x|^2$$

$$-\int_0^T\int_{-\infty}^0 g'(t-s)e^{-\omega(t-s)}|y(s)-y(t)|^2\,ds\,dt + \omega\int_0^T\int_{-\infty}^0 g(t-s)e^{-\omega(t-s)}|y(s)|^2\,ds\,dt$$

$$+\int_0^T (\Delta(\omega)\,I - 2A)y(t),y(t))\,dt = 2\int_0^T (e^{-\omega t}f(t),y(t))\,dt$$

Hence there exits M_T such that

$$\int_0^T |u(t)|^2\,dt \leq M_T \int_0^T |f(t)|^2 \tag{7.10}$$ est02

Consequently, we have

thm6.3 **Theorem 7.4.** *For $f \in L^2(0,T;H)$ there exists a unique solution u in $L^2(0,T;H)$ to (7.1) and it has the representation (7.5) and the energy identity (7.8).*

Proof: Since $C^2(0,T:H)$ is dense in $L^2(0,T;H)$ and $\mathcal{A}$ is closed, the theorem follows from (7.10).

corr6.2 **Corollary 7.1.** *Assume there exist a closed subspace V of H and $\delta > 0$ such that*

$$(A\phi,\phi) \leq -\delta\,|\phi|_V^2$$

Then, for $f \in L^2(0,T;H)$ there exists a unique solution u in $L^2(0,T;V)$ to (7.1) and it has the representation (7.5) and the energy identity

$$\int_0^t |u(s)|_V^2\,ds \leq \frac{1}{\delta}\int_0^t |f(s)|_{V^*}^2\,ds.$$

8 Dual System and Optimal Control Problems

In this section we consider the dual system to (1.6) [fde] and its application to a class of optimal control problems.

Let $X = H$ be a Hilbert space. For $u,\ p \in C^1(0,T;H)$ we have

$$
\begin{aligned}
&\int_0^T \left(\int_0^t g(t-s)u'(s)\,ds, p(t)\right)dt = \int_0^T \left(\int_s^T g(t-s)v(t)\,dt, u'(s)\right)ds \\
&= -(u(0), \int_0^T g(t)v(t)\,dt) - \int_0^T (u(s), \frac{d}{ds}\int_s^T g(t-s)p(t)\,dt)\,ds \\
&= (\int_0^T g(T-s)u(s)\,ds, p(T)) - (u(0), \int_0^T g(t)p(t)\,dt) - \int_0^T (u(s), \int_s^T g(t-s)p'(t)\,dt)\,ds.
\end{aligned}
\tag{8.1}
$$

dual

tm7 **Theorem 8.1.** *Let* $D_t^g : L^2(0,T;H) \to L^2(0,T;H)$ *be*

$$(D_t^g u)(t) = \int_0^t g(t-s)u'(s)\,ds$$

with domain

$$dom\,(D_t^g) = \{u \in L^2(0,T;H) : D_t^g u \in L^2(0,T;H),\ u(0^+) = 0\}$$

Then, D_t^g *is a densely defined, closed operator on* $L^2(0,T;H)$ *and the adjoint* $(D_t^g)^*$ *is given by*

$$((D_t^g)^* p)(s) = \frac{d}{ds}\int_s^T g(t-s)p(t)\,dt$$

with domain

$$
\begin{aligned}
dom\,((D_t^g)^*) = \{p \in L^2(0,T;H) : &\int_s^T g(t-s)p(t)\,dt \in H^1(0,T;X), \\
&(\int_s^T g(t-s)p(t)\,dt)(T^-) = 0\}.
\end{aligned}
$$

Proof: Since for $g^* \in L^1(0,T)$ is defined by (2.11) [gs] and from (2.12) [gs0]

$$u(t) = \int_0^t g^*(t-s)(D_t^g u)\,ds,$$

D_t^g is closed. For $y = (D_t^g)^* u$,

$$\int_0^T (y(t), u(t))\,dt = \int_0^T (\int_s^T y(t)\,dt, u'(s))\,ds$$

and

$$\int_0^T (D_t^g u, p(t))\,dt = \int_0^T (\int_s^T g(t-s)p(t)\,dt, u'(s))\,ds.$$

Since $u \in dom(D_t^g)$ are arbitrary,

$$\int_s^T g(t-s)p(t)\,dt = \int_s^T y(t)\,dt, \quad \text{for a.e. } s \in (0,T).$$

Hence, we obtain

$$\lim_{s\uparrow T}(\int_s^T g(t-s)p(t)\,dt)=0$$

and

$$y(s)=(D_t^g)^*p=\frac{d}{ds}\int_s^T g(t-s)p(t)\,dt.\square$$

Definition 7.1 (Weak solution) A function $u\in L^2(0,T;H)$ is a weak solution to (1.6) if

$$\int_0^T(-\int_s^T g(t-s)p'(t)\,dt+A^*p(s),u(s))+(p(s),f(s))\,ds-(u(0),\int_0^T g(t)p(t)\,dt)=0 \tag{8.2}$$

weak

for all $p\in C^1(0,T;H)\cap C(0,T;dom(A^*))$ satisfying $p(T)=0$.

Theorem 8.2. *Weak solutio to(1.6) is unique*

Proof: It follows from Theorem 7.4 that the dual system

$$\int_s^T g(t-s)p'(t)\,dt=A^*p(s)+f(s),\quad p(T)=0 \tag{8.3}$$

duals

has a solution $p\in C^1(0,T;H)\in C(0;T;dom(A^*)$ for all $f\in C^1(0,T;H)$ with $f(T)=0$. Since $\{f\in C^1(0,T;H)$ with $f(T)=0$ is dense in $L^2(0,T;H)$ the uniqueness follows from (8.2). $\square$

Consider the control problem

$$\min\quad\int_0^T(\ell(x(t))+h(u(t))\,dt, \tag{8.4}$$

cost

subject to (1.6);

$$\int_0^t g(t-s)x'(s)\,ds=Au(t)+Bu(t),\quad x(0)=x^0, \tag{8.5}$$

conts

where $x(t)\in X$, a Hilbert space and A is a maximal monotone operator in X Let U be a Hilbert space and $\hat{U}$ is a closed convex set in U and

$$u\in\mathcal{C}=\{u\in L^2(0,T;U):u(t)\in\hat{U},\,a.e.\}$$

denote the control function and $B\in\mathcal{L}(U,X)$. The functional ℓ and h are convex on X and U, respectively.

cont **Theorem 8.3.** *Problem (8.4)–(8.5) has an optimal control $u^*\in\mathcal{C}$.*

Proof: Since given $u\in\mathcal{C}$ (8.5) has a unique solution $x(\cdot;u)\in C(0,T;X)$. Thus, the optimal control problem is equivalent to minimizing

$$J(u)=\int_0^T(\ell(x(t;u))+h(u(t)))\,dt$$

over $u \in \mathcal{C}$. Suppose $u_n \in \mathcal{C}$ is a minimizing sequence of J over $\mathcal{C}$, i.e. $J(u_n) \downarrow$ and $\lim_{n\to\infty} J(u_n) = \eta = \inf_{u\in\mathcal{C}} J(u)$. Then there exists a weak convergent subsequence of (u_n, x_n) to (u^*, x^*) in $L^2(0,T;U \times X)$. Since $\mathcal{C}$ is weakly closed, $u^* \in \mathcal{C}$. From (8.2) it follows that x^* is a weak solution of (8.5) corresponding to u^*. Since convex functionals are weakly sequentially lower semi-continuous, $J(u^*) \le \eta$, i.e., $u^* \in \mathcal{C}$ is optimal. □

Let $\partial\ell(x^*)$ be the sub-differential of ℓ at $x^* \in X$, i.e.

$$\partial\ell(x^*) = \{\lambda \in X^* : \ell(x) - \ell(x^*) \ge (\lambda, x - x^*) \text{ for all } x \in X\}.$$

Define the Lagrange functional

$$L(x,u,p) = \int_0^T (\ell(x(s)) + h(u(s))\,ds + \int_0^T (p(s), Ax(s) + Bu(s) - (D_t^g x)(s))\,ds.$$

Theorem 8.4. *Let $u^* \in \mathcal{C}$ be an optimal to problem (8.4)–(8.5) and assume ℓ is C^1. Then,*

$$h(u) + (u, B^*p(t)) \ge h(u^*(t)) + (u^*(t), Bp(t)) \text{ for all } u \in \hat{U}, \quad \text{a.e. } t \in (0,T)$$

where the adjoint state satisfies

$$(D_t^g)^* p = A^* p(t) + \ell'(x^*(t)), \quad p(T) = 0. \tag{8.6}$$

Proof: Since $\ell'(x^*) \in C(0,T;X)$ there exists a unique solution to (8.6), Let $u = u^* + t\,(v - u^*) \in \mathcal{C}$ with $v \in \mathcal{C}$ and $t \in (0,1)$ and x be the corresponding solution of (8.5) to u. Note that

$$(\ell'(x^*), x - x^*) = ((D_t^g)^* p - A^* p, x - x^*) = (p, D_t^g(x - x^*) - A(x - x^*)) = (p, B(u - u^*)).$$

Then,

$$\begin{aligned} 0 \le J(u) - J(u^*) &= \int_0^T (E(x(s), x^*(s)) + (\ell'(x^*(s)), x(s) - x^*(s))\,ds \\ &+ \int_0^T (h(u(s)) - h(u^*(s)) + (p(t), B(u(s) - u^*(s)))\,ds \end{aligned} \tag{8.7}$$

where

$$E(x, x^*) = \ell(x) - \ell(x^*) - \ell'(x^*)(x - x^*).$$

Since

$$\frac{1}{t}\int_0^T E(x(s), x^*(s))\,ds \to 0 \text{ as } t \to 0^+$$

Since h is convex,

$$\int_0^T (h(u(s)) - h(u^*(s)) + (p(s), B(u(s) - u^*(s)))\,ds \le t\int_0^T (h(v(s)) - h(u^*(s)) + (p(s), B(v(s) - u^*(s))\,ds,$$

Now, since

$$\frac{1}{t}\int_0^T E(x(s), x^*(s))\,ds \to 0 \text{ as } t \to 0^+$$

letting $t \to 0^+$ in (8.7), we obtain

$$\int_0^T (h(v(s)) - h(u^*(s)) + (p(t), B(v(s) - u^*(s))\,ds \ge 0$$

for all $v \in \mathcal{C}$, which implies the necessary optimality. □

9 Case: $1 < \alpha < 2$

In this section we consider the case when $1 < \alpha < 2$;

$$D_t^\alpha u = \int_0^t g_{2-\alpha}(t-s)u''(s)\,ds = Au(t) + f(t),, \quad u(0 = u_0, \;\; u'(0) = v_0,$$

or in general

$$\int_0^t g(t-s)u''(s)\,ds = Au(t) + f(t), \quad u(0 = u_0, \;\; u'(0) = v_0. \tag{9.1}$$ a1-2

Equivalently, we have

$$\frac{d}{dt}u = v \quad \text{and} \quad \int_0^t g(t-s)v'(s)\,ds = Au(t) + f(t).$$

with

$$u(0) = u_0, \quad v(0) = v_0.$$

Assume $-A$ is self-adjoint and positive on a Hilbert space H and define

$$V = dom((-A)^{\frac{1}{2}}) \text{ with } |\phi|_V^2 = \langle -A\phi, \phi\rangle.$$

Define a linear operator $\mathcal{A}$ on $X = V \times L^2_g(-\infty, 0; H)$ by

$$\mathcal{A}(u, z) = (z(0), z')$$

with

$$dom(\mathcal{A}) = \{(u, z) \in X : \mathcal{A}(u, z) \in X \text{ and } \int_{-\infty}^0 g(\theta)z'(\theta) = Au\}.$$

a12 **Theorem 9.1.** *The linear operator $\mathcal{A}$ is m-dissipative and $\mathcal{A}$ generates a C_0 semigroup $T(t)$ on X. $u(t) = (T(t)(u_0, v_0))_1 \in C(0, T; V)$ defines a mild solution to (a1-2 12.4).*

Proof: For $(u, z) \in dom(\mathcal{A})$

$$(\mathcal{A}(u, z), (u, z)) = (-Au, z(0)) + \textstyle\int_{-\infty}^0 g(\theta)(z'(\theta), z(\theta))\,d\theta$$

$$= \int_{-\infty}^0 g(\theta)(z'(\theta), z(\theta) - z(0))\,d\theta.$$

From (sta0 2.1)

$$(\mathcal{A}(u, z), (u, z)) = -\lim_{\epsilon \to 0^+} \int_{-\infty}^0 \frac{g(\theta) - g(\theta - \epsilon))}{\epsilon} |z(\theta) - z(0)|^2\,d\theta,$$

and thus

$$(\mathcal{A}(u, z), (u, z)) = -\int_{-\infty}^0 g'(\theta)|z(\theta) - z(0)|^2\,d\theta \leq 0.$$

For the resolvent

$$\lambda\,(u, z) - \mathcal{A}(u, z) = (f^1, f^2)$$

is equivalent to

$$\lambda\, z - z' = f^2, \quad \lambda\, u - z(0) = f^1.$$

From the first equation

$$z(\theta) = e^{\lambda\,\theta} z(0) + \int_\theta^0 e^{\lambda(\theta-\xi)} f^2(\xi)\, d\xi \tag{9.2}$$

Res0

and

$$\lambda(\int_{-\infty}^0 e^{\lambda\theta} g(\theta)\, d\theta) z(0) + \int_{-\infty}^0 g(\theta)(\lambda \int_\theta^0 e^{\lambda(\xi-\theta)} f^2(\xi)\, d\xi - f^2(\theta))\, ds = Au.$$

From the second equation

$$u = (\lambda\Delta(\lambda)\, I - A)^{-1}(\Delta(\lambda) f^1 + \int_{-\infty}^0 g(\theta)(f^2(\theta) - \lambda \int_\theta^0 e^{\lambda(\theta-\xi)} f^2(\xi)\, d\xi)\, d\theta) \tag{9.3}$$

Res1

and

$$z(0) = \lambda\, u - f^1. \tag{9.4}$$

Res2

Thus, $R(\lambda\, I - \mathcal{A}) = X$. □

It follows from (9.3) that $u(t) = S_1(t)u + S_2(t)v$ with

$$\begin{aligned} S_1(t)u &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} (\lambda\Delta(\lambda)\, I - A)^{-1} \Delta(\lambda) u\, d\lambda \\ S_2(t)v &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} (\lambda\Delta(\lambda)\, I - A)^{-1} \frac{\Delta(\lambda)}{\lambda} v\, d\lambda. \end{aligned} \tag{9.5}$$

sol2

Assume A is a sectorial operator, i.e., there exist $M > 0,\ \theta_0 > 0$ such that

$$|(\lambda\, I - A)^{-1}| \le \frac{M}{|\lambda|} \quad \text{on } \Sigma_{\theta_0} = \{\lambda \in C : \arg(\lambda) \le \frac{\pi}{2} + \theta_0\} \cap \{\lambda \ne 0\}$$

Assume that if $\lambda \in \Sigma_{\theta_0}$, then $\lambda\Delta(\lambda) \in \Sigma_{\tilde{\theta}}$ for $\tilde{\theta} > 0$. Let $\Gamma_{\theta,\delta}$ be the integration path defined by

$$\Gamma^\pm = \{z \in C : |\lambda| \ge \delta,\ arg(\lambda) = \pm(\frac{\pi}{2} + \lambda)\}, \quad \Gamma_0 = \{z \in C : |z| = \delta,\ |arg(z)| \le \frac{\pi}{2} + \theta\}$$

for some $\delta > 0$ and $0 < \theta \le \tilde{\theta}$. Then, the solution map $S_1(t)$ and $S_2(t)$ is given by

$$S_1(t)x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda\Delta(\lambda)\, I - A)^{-1} \Delta(\lambda) x\, d\lambda,$$

$$S_2(t)x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda\Delta(\lambda)\, I - A)^{-1} \Delta(\lambda) \lambda^{-1} x\, d\lambda,$$

using the Cauchy integral representation (9.5) and the analytic continuation. Let $P(t) \in \mathcal{L}(X),\ t > 0$ be

$$P(t)x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda\Delta(\lambda)\, I - A)^{-1} x\, d\lambda.$$

Then

$$\frac{d}{dt}S_1(t) = P(t)A$$

and the solution $u(t)$ to (12.4) is given by

$$u(t) = S_1(t)u_0 + S_2(t)v_0 + \int_0^t P(t-s)f(s)\,ds$$

for $u_0,\ v_0 \in X$ and $f \in C(0,T;X)$.

10 Finite Difference Method

In this section we develop the finite difference scheme for (1.6) and analyze the stability and the convergence of the scheme for a general class of (1.6) with maximal monotone operators A.

Let $h > 0$ be a stepsize and define

$$g_j = \int_{-(j+1)h}^{-jh} g(s)\,ds, \quad j \le 0$$

The sequence $\{u_j^k,\ j \le 0\}$ approximating $u(kh, -jh) \in X$ is generated by

$$g_0\frac{u_0^k - u_{-1}^k}{h} + \sum_{j=-1}^{-(k-1)} g_j\,\frac{u_j^k - u_{j-1}^k}{h} = Au_0^k + f^k \tag{10.1}$$

dif

$$u_{j-1}^k = u_j^{k-1}, \quad j \le 0, \quad u_j^k = x \text{ for } j \le -k.$$

That is,

$$\frac{u_j^k - u_j^{k-1}}{h} = \frac{u_j^k - u_{j-1}^k}{h} \tag{10.2}$$

dif1

approximates $\dfrac{d}{dt}u(t+\cdot) = \mathcal{A}u(t+\cdot)$. Note that

$$u_j^k = u_0^{k+j} = u^{k+j}.$$

Thus, (10.1) is equivalent to

$$g_0\frac{u^k - u^{k-1}}{h} + \sum_{j=-1}^{-(k-1)} g_j\,\frac{u^{k+j} - u^{k+j-1}}{h} = Au^k + f^k \tag{10.3}$$

dif0

which is an approximation for (1.6) directly. We have the following stability results for (10.1);

stab1 **Theorem 10.1.** *If A is dissipative, for $f = 0$ we have $|u_0^k| \le |x|$. In general we assume that for all $u \in dom(A)$, there $u^* \in F(u)$ such that*

$$\langle Au, u^*\rangle \le -\delta\,|u|^2, \tag{10.4}$$

cond1

then

$$\max_{0\le k\le N} |u^k| \le \frac{1}{\delta} \max_{0\le k\le N} |f^k|.$$

Proof:For $f=0$ since

$$\sum_{j=0}^{-(k-1)} g_j \frac{u_j-u_{j-1}}{h} = -\sum_{j=0}^{-(k-2)} \frac{g_{j-1}-g_j}{h}(u_{j-1}-u_0), \tag{10.5}$$

dif1

it follows from (10.1) that for $u^* \in F(u_0^k)$

$$\langle Au_0^k, u^*\rangle = -\sum_{j=0}^{-(k-2)} \frac{g_{j-1}-g_j}{h}((u_{j-1}^k,u^*)-|u_0^k|^2).$$

Suppose $|u_0|>|u_j|,\ j<0$ then

$$\langle Au_0, u^*\rangle > \sum_{j=0}^{-(k-2)} \frac{g_j-g_{j-1}}{h}(|u_0|^2-|u_{j-1}^k||u_0|)>0$$

which contradicts to the fact that A is dissipative. Thus, $\max_{j\le 0}|u_j^k| \le \max_{j\le 0}|u_j^{k-1}|$.

In general, suppose $|u^k|\ge|u^j|$ for $0\le j\le N$. From (10.5)

$$-\langle Au_0^k, u^*\rangle \le |u_0^k||f^k|$$

for $u^*\in F(u_0^k)$ and thus from (??)

$$|u_0^k| \le \frac{1}{\delta}\,|f^k|.$$

Hence we obtain

$$\max_{0\le k\le N}|u^k| \le \frac{1}{\delta}\max_{0\le k\le N}|f^k|.$$

Assume $X=H$ is a Hilbert space.

Theorem 10.2. *Assume for $\delta>0$*

$$(A\phi,\phi)\le -\delta|\phi|^2, \quad \phi\in dom(A).$$

For all $k\ge 1$

$$\sum_{j=0}^{-(k-1)} g_j|u_j^k|^2+\sum_{\ell=1}^{k}\sum_{j=0}^{-(\ell-2)}\frac{g_j-g_{j-1}}{h}|u_j^\ell-u_0^\ell|^2h+\delta\sum_{\ell=0}^{k}|u_0^\ell|^2\,h \le \sum_{j=0}^{-(k-1)} g_j|x|^2+\frac{1}{\delta}\sum_{\ell=1}^{k}|f^k|^2\,h. \tag{10.6}$$

est2

Proof: From (10.5)

$$(\frac{u_j^k-u_j^{k-1}}{h},\frac{1}{2}(u_j^k+u_j^{k-1}))=\frac{1}{2}(|u_j^k|^2-|u_j^{k-1}|^2)=(\frac{u_j^k-u_{j-1}^k}{h},\frac{1}{2}(u_j^k+u_{j-1}^k)),$$

where

$$\sum_{j=0}^{-(k-1)} g_j(\frac{u_j^k-u_{j-1}^k}{h},\frac{1}{2}(u_j^k+u_{j-1}^k)-u_0^k)=\frac{1}{2}\sum_{j=0}^{-(k-2)}\frac{g_{j-1}-g_j}{h}|u_j^{k+1}-u_0^{k+1}|^2\,h.$$

Since from (dif 10.1)

$$\sum_{j\le 0} g_j(\frac{u_j^k-u_{j-1}^k}{h},u_0^k)=(Au_0^k+f^k,u_0^k),$$

summing this over k we obtain

$$\sum_{j=0}^{-(k-1)} g_j|u_j^k|^2+\sum_{\ell=1}^k\sum_{j=0}^{-(\ell-2)}\frac{g_j-g_{j-1}}{h}|u_j^\ell-u_0^\ell|^2h=\sum_{j=0}^{-(k-1)} g_j|x|^2+2\sum_{\ell=1}^k(Au_0^\ell+f^\ell,u_0^\ell)\,h.$$

Hence, from the assumption we obtain the desired estimate. □

Now, we have the convergence results;

conv1 **Theorem 10.3.** *We assume*

$$\langle Ax_1-Ax_2,x^*\rangle\le -\delta\,|x_1-x_2|^2. \qquad (10.7)$$ dif2

for some $x^*\in F(x_1-x_2)$. *Define the linear interpolation*

$$U_h(t)=u(kh)+\frac{t-kh}{h}(u(kh)-u((k-1)h))\quad \textit{if } t\in(k-1)h,kh].$$

Then,

$$|u_h-U_h|_{L^2(0,T;H)}\to 0\quad \textit{as } h\to 0^+.$$

Proof: If we let $U_j^k=U((k-j)h)=u((k-j)h)$, then we have

$$g_0\frac{U_0^k-U_{-1}^k}{h}+\sum_{j=0}^{-(k-1)} g_j\,\frac{U_j^k-U_{j-1}^k}{h}=AU_0^k+f^k+E^k \qquad (10.8)$$ form

where

$$E^k=\int_{-kh}^0 g(\theta)(u'(kh+\theta)-U'(kh\theta))\,d\theta\to 0$$

Let $X=H$ be a Hilbert space. The, we have

$$\sum_{\ell}^N|E^\ell|^2\,h\to 0\quad \text{as } h\to 0^+$$

If we define

$$u_h(t)=u^k+\frac{t-kh}{h}(u^k-u^{k-1})\quad \text{if } t\in(k-1)h,kh]$$

then it follows from (est2 10.6) and (dif2 10.7) that

$$|u_h-U_h|_{L^2(0,T;H)}\to\frac{1}{\delta}\,|E_h|_{L^2(0,T;H)}\to 0$$

as $h\to 0^+$. □

In general, let X be a Banach space and then we have;

Corollary 10.1.

$$|u_h-U_h|_{C(0,T;X)}\to 0,\quad \textit{as } h\to 0^+,$$

assuming (dif2 10.7) *and* $f\in C(0,T;X)$.

Proof: Using the same arguments as above, from (dif2 10.7) and (form 10.8) we have

$$|u_h-U_h|_{C(0,T;X)}\le\frac{1}{\delta}\,|E_h|_{C(0,T;X)}\to 0.\square$$

10.1 Cone invariance and Maximum Principle

Let $H = L^2(\Omega)$ and $\mathcal{C}$ be a a closed cone in H. A is cone preserving, i.e.,

$$(I - s\,A)^{-1}\mathcal{C} \subset \mathcal{C}$$

for all sufficiently small $s > 0$. Since

$$u^k = (I - \frac{h}{g_0}A)^{-1}(\sum_{j=0}^{-(k-2)} \frac{g_j - g_{j-1}}{g_0} u^{k+j} + \frac{hg_{-(k-1)}}{g_0} u^0 + f^k)$$

By induction in k, we have $u^k \in \mathcal{C}$ if $u^0 \in \mathcal{C}$ and $f^k \in \mathcal{C}$.

Let $\mathcal{C} = \{\phi \in H : \phi \geq 0 \text{ a.e. }\}$ Then, $u^k \geq 0$ a.e. if $u^0 \geq 0$ and $f^k \geq 0$.

11 Semi-linear equations

In this section we consider the semilinear equation

$$D_t^\alpha x = Ax(t) + F(x(t)), \quad x(0) = x_0$$

or equivalently

$$x(t) = S(t)x_0 + \int_0^t P(t-s)F(x(s))\,ds, \ t \geq 0 \tag{11.1}$$

Lip

with the locally Lipschitz function F in X. Assume

$$|S(t)| \leq \psi(t) = C\min(1, t^{-\alpha}), \quad |P(t)| \leq C\min(t^{-1+\alpha}, t^{-1}).$$

and that $F(0) = 0$ and

$$|F(x) - F(y)| \leq \rho(M)\,|x - y| \quad \text{for} \quad |x|,\ |y| \leq M. \tag{11.2}$$

cond

For given $f \in C(0,T;X)$ consider the map from $C(0,\infty,X)$ to $C(0,\infty;X)$ by

$$(\Psi x)(t) = f(t) + \int_0^t P(t-s)F(x(s))\,ds \tag{11.3}$$

fixed

First, we establish the local existence of solutions.

Theorem (Local Existence) Assume $|f(t)| \leq M_0(1+t^\alpha)$. Then, there exists a $\tau > 0$ such that (fixed 11.3) has a fixed point $x \in C(0,\tau;X)$.

Proof: For $|x(t)| \leq M$ on $[0,\tau]$ for some $M \geq 2M_0$ we have

$$|\int_0^t P(t-s)F(x(s))\,ds| \leq \int_0^t C(t-s)^{-1+\alpha} M\rho(M)\,ds \leq CM\rho(M)\,\frac{t^\alpha}{\alpha}$$

Let $\tau = \tau(M_0)$ is chosen so that

$$M_0(1+\tau^\alpha) + CM\rho(M)\,\frac{\tau^\alpha}{\alpha} \leq M, \tag{11.4}$$

tau

given $M_0 > 0$. It follows from (cond 11.5)

$$|(\Psi x_1)(t) - (\Psi x_2)(t)| \leq C\rho(M)\frac{\tau^\alpha}{\alpha}|x_1 - x_2|_{C(0,\tau;X)}$$

for $|x_1|,\ |x_2| \leq M$. Thus, Ψ has a unique fixed point $x \in C(0,\tau;X)$ satisfying $|x(t)| \leq M,\ t \in [0,\tau]$, provided that (tau 11.4) holds, since $C\rho(M)\frac{\tau^\alpha}{\alpha} < 1$. $\square$

Hence, given $t \geq 0$ for $h \geq 0$

$$x(t+h) = S(t+h)x_0 + \int_0^t P(t+h-s)F(x(s)) + \int_t^{t+h} P(t+h-s)F(x(s))\,ds$$

$$= f(t+h) + \int_0^h P(h-\sigma)F(x(t+\sigma))\,d\sigma$$

has a unique solution $x(t+h),\ 0 \leq h \leq \tau$ as a fixed point to (fixed 11.3) with

$$f(t+h) = S(t+h)x_0 + \int_0^t P(t+h-s)F(x(s))\,ds,;\ h \geq 0$$

provided that $f(t+h) \leq M_0(1+h^\alpha)$ and some M_0.

Next, we establish a priori bound of $x(t)$. Assume $x \in M\,\min(1,t^{-\alpha})$ for some $M > 0$ and that

$$\rho(s)s = s^\gamma.$$

We will use

$$\int_0^t (t-s)^{-1+\delta}s^{-\delta}\,ds = \frac{1}{\Gamma(1-\delta)\Gamma(\delta)}$$

for $0 < \delta < 1$. Let

$$I(t) = |\int_0^t P(t-s)F(x(s)\,ds|$$

For $t \leq 1$

$$I(t) \leq M^\gamma \int_0^t C(t-s)^{-1+\alpha}\,ds = \frac{CM^\gamma}{\alpha}.$$

For $1 \leq t \leq 2$

$$I(t) \leq \int_1^t C(t-s)^{-1+\alpha}M^\gamma s^{-\gamma\alpha}\,ds + \int_0^1 CM^\gamma(t-s)^{-1+\alpha}\,ds$$

$$\leq CM^\gamma\,[(t-1)^\alpha - t^\alpha + \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}t^{-\alpha(\gamma-1)}].$$

For $t \geq 2$

$$I(t) \leq \int_{t-1}^t C(t-s)^{-1+\alpha}M^\gamma s^{-\gamma\alpha})\,ds + \int_1^{t-1} C(t-s)^{-1}M^\gamma s^{-\gamma\alpha}\,ds + \int_0^1 C(t-s)^{-1}M^\gamma\,ds$$

$$\leq CM^\gamma\,[(t-1)^{-\gamma\alpha}t^\alpha + \tfrac{1}{\Gamma(\delta)\Gamma(1-\delta)}M^\gamma + (t-1)^{-1}t^\alpha]t^{-\alpha}$$

with $\delta = \alpha(\gamma-1)$. It thus follows that there exists β independent of $t \geq 0$ such that

$$I(t) \leq \beta CM^\gamma\,\min(1,t^{-\alpha})$$

It can be proved that if $M_0 > 0$ is sufficiently small there exists $M > 0$ such that

$$M_0 + \beta C M^\gamma \leq M. \tag{11.5}$$ cond

Theorem (Asymptotic Stability) Assume $\rho(s)s = s^\gamma$ and condition (11.5) holds. Then, (11.1) has a unique solution in $C(0,\infty;X)$ satisfying $|x(t)| \leq M\, \min(1, t^{-\alpha})$.

Proof: Using exactly the same arguments as above, without loss of the generality with same $\beta > 0$ we have

$$|\Psi x_1 - \Psi x_2| \leq \beta C M^{\gamma-1}|x_1 - x_2|.$$

The claim follows since $\beta C M^{\gamma-1} < 1$. $\square$

11.1 Sectorial Case

Assume there exits $\alpha_1,\ \alpha_2 \geq 0$ such that

$$|A^{-\alpha_1}(F(x) - F(y))| \leq \rho(M)\,|A^{\alpha_2}(x-y)| \quad \text{for} \quad |A^{\alpha_2}x|,\ |A^{\alpha_2}y| \leq M. \tag{11.6}$$ cond1

and

$$|A^{-\alpha_1}F(0)|_X \leq c. \tag{11.7}$$ cond2

with $0 \leq \alpha_1 + \alpha_2 < 1$. Let $X_{\alpha_2} = dom(A^{\alpha_2})$. For $u \in C([0,\tau], X_{\alpha_2})$ define the map

$$(\Psi(u))(t) = S(t)x_0 + \int_0^t P(t-s)F(u(s))\,ds.$$

Theorem 11.1. *(Local Existence) For $x_0 \in X_{\alpha_2}$ there exists a $\tau > 0$ such (11.1)has a unique solution $u \in C(0,\tau;X_{\alpha_2})$.*

Proof: For $u(0,\tau,X_{\alpha_2})$ satisfying $|u| \leq M$ on $[0,\tau]$ it follows from Corollary 5.4 that

$$|A^{\alpha_2}\Psi(u)| = |S(t)||A^{\alpha_2}x_0| + |\int_0^t A^{\alpha_1+\alpha_2}P(t-s)A^{-\alpha_1}F(u(s))\,ds|.$$

$$\leq |A^{\alpha_2}x| + \int_0^t C(t-s)^{-\beta}(\rho(M)M + c)\,ds$$

$$\leq |A^{\alpha_2}x| + (\frac{C}{1-\beta}\tau^{1-\beta}(\rho(M)M + c)$$

where $\beta = \alpha\,(1 - (\alpha_1 + \alpha_2))$. Let $\tau = \tau(M_0)$ such that

$$M_0 + \frac{C}{1-\beta}\tau^{1-\beta}(\rho(M)M + c) \leq M$$

given $M_0 = |A^{\alpha_2}x_0|$. It follows from (11.6) that

$$|A^{\alpha_2}(\Psi(u_1) - \Psi(u_2))| \leq \frac{C}{1-\beta}\tau^{1-\beta}\rho(M)\,|A^{\alpha_2}(u_1 - u_2)|$$

Thus, Ψ has a unique fixed point in $x \in C(0,\tau,X_{\alpha_2})$ satisfying $|A^{\alpha_2}x(t)| \leq M$ on $[0,\tau]$, which defines a solution to (11.1) $\square$

12 Examples

In this section we discuss the application of our theory for concrete examples.

12.1 Fractional Parabolic equations

Let Ω is a bounded open set in R^d. Consider the fractional parabolic equation

$$D_t^\alpha u = \nabla \cdot (a(x)\nabla u) + b(x) \cdot \nabla u + f(x,u), \tag{12.1}$$

parab

where $u \in C([0,\tau] \times \Omega)$ and $f : R^d \times R \to R$ is locally Lipschitz function. Define the second order elliptic operator

$$Au = \nabla(a(x)\nabla u) + b(x) \cdot \nabla u$$

with $a \in R^{d\times d} \in C^1(\Omega)$ is symmetric and positive definite and $b \in C(\Omega)$. The linear operator A with

$$dom\,(A) = \{u \in C^2(\Omega) \cap H_0^1(\Omega)\}$$

is dissipative in $X = C(\Omega)$. In fact, if $u(\zeta_0) \geq |u|$ for $\zeta_0 \in \Omega$, then $(\nabla u)(\zeta_0) = 0$ and $H(x_0) \leq 0$. and

$$(Au)(\zeta_0) = \operatorname{tr} a(\zeta_0)H(\zeta_0) + (\nabla \cdot a + b) \cdot (\nabla u)(\zeta_0) \leq 0$$

where $H_{i,j} = u_{x_i,x_j}$ is the Hessian of u. Similarly, if $u(\zeta_0) \leq |u|$, then $(Au)(\zeta_0) \geq 0$. Define the nonlinear operator by

$$(F(u))(x) = f(x,u(x))$$

We assume $f(0) = 0$ and

$$|f(x) - f(y)| \leq \rho(M)\,|x-y| \quad \text{for} \quad |x|,\ |y| \leq M. \tag{12.2}$$

assm

Then, ([cond]11.5) is satisfied and it follows from Theorem that ([parab]12.1) has a local in time solution $u \in C([0,\tau] \times \Omega)$. Moreover, if $f(x,u)u \leq 0$, then the solution u is global in time and $|u(t)|_X \leq |u_0|_X$.

12.2 Fractional Scalar conservation law

In this section we consider the scalar conservation law

$$D_t^\alpha u + (f(u))_x + f_0(x,u) = 0, \;\; t > 0 \quad u(x,0) = u_0(x),\; x \in R^d \tag{12.3}$$

fcon

where $f : R \to R^d$ is C^1. Let $X = L^1(R^d)$ and define

$$Au = -(f(u))_x,$$

where we assume $f_0 = 0$ for the sake of simplicity of our presentation. Define

$$\mathcal{A}\phi = \phi'(\theta)$$

with domain

$$dom(\mathcal{A}) = \{\int_{-\infty}^0 g(\theta)\phi'(\theta)\,d\theta = A\phi(0)\}.$$

Let

$$\mathcal{C} = \{\phi \in Z : \phi \geq 0\}.$$

Since $Ac = 0$ for all constant c, it follows that

$$\phi - c \in \mathcal{C} \Longrightarrow (I - \lambda\,\mathcal{A})^{-1}\phi - c \in \mathcal{C}.$$

Similarly,

$$c - \phi \in \mathcal{C} \Longrightarrow c - (I - \lambda\,\mathcal{A})^{-1}\phi \in \mathcal{C}.$$

Thus, without loss of generality, one can assume f is bounded. Let $\rho \in C^2(R)$ be a monotonically increasing function satisfying $\rho(0) = 0$ and $\rho(x) = sgn(x),\ |x| \geq 1$. Note that

$$-(f(u_1)_x - f(u_2)_x, \rho(u_1 - u_2)) = (f(u_1) - f(u_2), \rho'(u_1 - u_2)\,(u_1 - u_2)_x),$$

$$= (\eta, \rho'(u_1 - u_2)\,(u_1 - u_2)_x(u_1 - u_2)),$$

where

$$\eta = \int_0^1 f_u(u_2 + \tau\,(u_1 - u_2))\,d\tau.$$

If we define $\Psi(x) = \int_0^x \sigma\rho'(\sigma)\,d\sigma$, then

$$(\eta\,(u_1 - u_2), \rho'(u_1 - u_2)\,(u_1 - u_2)_x) = -(\Psi(u_1 - u_2), \eta_x)$$

where $u_\tau = u_2 + \tau\,(u_1 - u_2)$ and

$$\eta_x = \int_0^1 (f_{xu}(s, x, u_\tau) + f_{uu}(s, x, u_\tau)(u_\tau)_x)\,d\tau.$$

Define $\rho_\epsilon(x) = \rho(\frac{x}{\epsilon})$ for $\epsilon > 0$. Then

$$|(\eta\,(u_1 - u_2), \rho_\epsilon'(u_1 - u_2)\,(u_1 - u_2)_x)| = \epsilon\,(\Psi(\frac{u_1 - u_2}{\epsilon}), \eta_x) \leq const\,\epsilon\,|\eta_x|_1 \to 0$$

as $\epsilon \to 0$. Note that for $u \in L^1(R^d)$

$$(u, \rho_\epsilon(u)) \to |u| \quad \text{and} \quad (\psi, \rho_\epsilon(u)) \to (\psi, sgn_0(u)) \ \text{ for } \ \psi \in L^1(R^d)$$

as $\epsilon \to 0^+$. Thus,

$$\langle Au_1 - Au_2, sgn_0(u_1 - u_2)\rangle \leq 0$$

and A is monotone. It is show in [2] that

$$range(\lambda\,I - A) = X,$$

i.e., for any $g \in X$ there exists an entropy solution satisfying

$$(sign(u - k)(\lambda u - g), \psi) \leq (sing(u - k)(f(u) - f(k)), \psi_x)$$

for all $\psi \in C_0^1(R^d)$ and $k \in R$. Hence A has a maximal monotone extension in $L^1(R^d)$.

12.3 Fractional Hamilton-Jacobi equation

Let u is a solution to a scalar conservation in R^1, then $v=\int^x u\,dx$ satisfies the fractional Hamilton-Jacobi equation

$$D_t^\alpha v+f(v_x)=0.$$

Let $X=C_0(R^d)$ and

$$Av=-f(v_x)\quad dom(A)=\{f(v_x)\in X\}$$

Then, for $v_1,\ v_2\in C^1_(R^d)$

$$\langle A(v_1-v_2),\delta_{x_0}\rangle=-(f((v_1)_x(x_0))-f((v_2)_x(x_0)))=0$$

where $x_0\in R^n$ such that $|v|_X=|v(x_0)|$. It also can be proved IK [2] that

$$range(\lambda\,I-A)=X\quad\text{for }\lambda>0.$$

That is, there exists a unique viscosity solution to $\lambda v-f(v_x)=g$; for all $\phi\in C^1(\Omega)$ if $v-\phi$ attains a local maximum at $x_0\in R^d$, then

$$\lambda\,v(x_0)-g(x_0)+f(\phi_x(x_0))\leq 0$$

and if $v-\phi$ attains a local minimum at $x_0\in R^d$, then

$$\lambda\,v(x_0)-g(x_0)+f(\phi_x(x_0))\geq 0.$$

12.4 Fractional Semilinear wave equation

In this section we consider the fractional semilinear wave equation of the form;

$$D_t\alpha u'(s)\,ds=A_0u(t)+F(u(t)),\quad u(0=u_0,\ \ u'(0)=v_0. \tag{12.4}$$ a1-2

Let $-A_0$ be a positive self-adjoint operator on a Hilbert space H. $V=dom(-(A_0)^{\frac{1}{2}})$

12.5 Fractional Navier Stokes

In this section we discuss a fractional incompressible Navier Stokes equation

$$\begin{aligned}&D_t^\alpha u+u\cdot\nabla u+\operatorname{grad}p=\nu\,\Delta u\\&\operatorname{div}u=0\end{aligned} \tag{12.5}$$ NS

where u is the velocity field defined on domain Ω with Lipschitz boundary Γ and p is the pressure. Let V be the divergence free closed subspace of $H^1_0(\Omega)^d$:

$$V=\{u\in H^1_0(\Omega)^d:\operatorname{div}u=0\}$$

and X be the closure of V with respect to $L^2(\Omega)^d$ norm:

$$X=\{u\in H^1_0(\Omega)^d:\operatorname{div}u=0,\ n\cdot=0\text{ at }\Gamma\}$$

Let P be the orthogonal projection of $L^2(\Omega)^d$ onto X and define the Stokes operator $-A$ by

$$-Au = P\Delta u \text{ with } dom\,(A) = H^2(\Omega)^d \cap V$$

and the convection

$$F(u) = -P(u \cdot \nabla u) \text{ for } u \in V$$

Then, (12.5) is equivalently written as

$$D_t^\alpha u = Au + F(u).$$

The Stokes operator $-A$ is positive self-adjoint operator X with $dom\,((-A)^{1/2}) = V$. Moreover,

$$|F(u) - F(v)|_{V_{-1/2}} \le c\,|u - v|_V(|u|_V + |v|_V),$$

where $V_{1/2} = dom((-A)^{1/4}$ and thus (11.6) is satisfied with $\alpha_1 = \frac{1}{4},\ \alpha_2 = \frac{1}{2}$. It follows from Theorem that (12.5) has a local solution $u \in C(0, \tau, V)$ in time, satisfying

$$u(t) = S(t)x + \int_s^t P(t-s)F(u(s))\,ds.$$

13 Space varying model

In this section we consider space varying cases;
<u>Case 1 (Space varying fraction $0 \le \alpha(x) < 1$)</u>

$$\int_0^t \frac{u'(x,s)}{(t-s)^{-\alpha(x)}}\,ds \in Au(x,t) + f(t). \tag{13.1}$$ fdes1

<u>Case 2 (Space varying weight $0 \le \tilde{g}(x,s)$)</u>

$$\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} u'(x,s)\,ds + \int_0^t \tilde{g}(x,t-s)u'(x,s)\,ds \in Au(x,t) + f(t). \tag{13.2}$$ fdes2

Assume X is a Banach space and $A \subset X \times X$ is dissipative. Let $Z = C(-\infty, 0]; X)$ and $\mathcal{A}z = z'(\theta)$ in Z For Case 1 define $g(x,\theta) = |\theta|^{-\alpha(x)}$ for $x \in \Omega$ and $\theta \in (0, -\infty)$. For Case 2 assume $s \to g(x,s)$ is decreasing for every $x \in \Omega$ and set $g(x,\theta) = g_\alpha(|\theta|) + \tilde{g}(x,|\theta|)$ Note that $\theta \to g(x,\theta)$ is monotonically increasing for every $x \in \Omega$. Define

$$dom(\mathcal{A}) = \{z' \in Z,\ z(0) \in dom(A) \text{ and } \int_{-\infty}^0 g(x,\theta)z'(x,\theta) \in Az(0)$$

Using exactly the same arguments as for Theorem 3.1, $\mathcal{A}$ is maximal monotone in Z and generate a nonlinear semigroup on Z. Thus Both (13.1) and (13.2) has a unique mild solution $u \in C(0,T;X)$.

14 Fractional equations in Space

In this section we consider the nonlocal diffusion equation of the form

$$u_t = Au = \int_{R^d} J(z)(u(x+z) - u(x))\,dz.$$

Or, equivalently

$$Au = \int_{(R)^d)^+} J(z)(u(x+z) - 2u(x) + u(x-z))\,dz$$

for the symmetric kernel J in R. It will be shown that

$$(Au, \phi)_{L^2} = \int_{R^d}\int_{(R)^d)^+} J(z)(u(x+z) - u(x))(\phi(x+z) - \phi(x))\,dz\,dx$$

and thus A has a maximum extension.

Also, the nonlocal Fourier law is given by

$$Au = \nabla\cdot(\int_{R^d} J(z)\nabla u(x+z)\,dz).$$

Thus,

$$(Au, \phi)_{L^2} = \int_{R^d\times R^d} J(z)\nabla u(x+z)\cdot\nabla\phi(x)\,dz\,dx$$

Under the kernel J is completely monotone, one can prove that A is a maximal monotone extension.

14.1 Jump diffusion Model for American option

In this section we discuss the American option for the jump diffusion model

$$u_t + (x - \frac{\sigma^2}{2})u_x + \frac{\sigma^2 x^2}{2}u_{xx} + Bu + \lambda = 0, \quad u(T,x) = \psi,$$

$$(\lambda, u-\psi) = 0, \quad \lambda \leq 0, \quad u \geq \psi$$

where the generator B for the jump process is given by

$$Bu = \int_{-\infty}^{\infty} k(s)(u(x+s) - u(x) + (e^s - 1)u_x)\,ds.$$

The CMGY model for the jump kernel k is given by

$$k(s) = \begin{cases} Ce^{-M|s|}|s|^{1+Y} = k^+(s) & s \geq 0 \\ Ce^{-G|s|}|s|^{1+Y} = k^-(s) & s \leq 0 \end{cases}$$

Since

$$\int_{-\infty}^{\infty} k(s)(u(x+s) - u(x)\,ds = \int_0^{\infty} k^+(s)(u(x+s) - u(x))\,ds + \int_0^{\infty} k^-(s)(u(x-s) - u(x))\,ds$$

$$= \int_0^{\infty} \frac{k^+(s) + k^-(s)}{2}(u(x+s) - 2u(x) + u(x-s))\,ds + \int_0^{\infty} \frac{k^+(s) - k^-(s)}{2}(u(x+s) - u(x-s))\,ds.$$

Thus,

$$\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} k(s)(s)(u(x+s)-u(x)\,ds)\phi\,dx$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} k_s(s)(u(x+s)-u(x))(\phi(x+s)-\phi(s))\,ds\,dx$$

$$+\int_{-\infty}^{\infty}(\int_{0}^{\infty} k_u(s)(u(x+s)-u(s)))\phi(x)\,dx$$

where

$$k_s(s)=\frac{k^+(s)+k^-(s)}{2},\quad k_u(s)=\frac{k^+(s)-k^-(s)}{2}$$

and hence

$$(Bu,\phi)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_s(s)(u(x+s)-u(x))(\phi(x+s)-\phi(s))\,ds\,dx$$

$$+\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} k_u(s)(u(x+s)-u(s)))\phi(x)\,dx+\omega\int_{-\infty}^{\infty} u_x\phi\,dx.$$

where

$$\omega=\int_{-\infty}^{\infty}(e^s-1)k(s)\,ds.$$

If we equip $V=H^1(R)$ by

$$|u|_V^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_s(s)|u(x+s)-u(x)|^2\,ds\,dx+\frac{\sigma^2}{2}\int_{-\infty}^{\infty}|u_x|^2\,dx,$$

then $A+B\in\mathcal{L}(V,V^*)$ and $A+B$ generates the analytic semigroup on $X=L^2(R)$.

14.2 Numerical approximation of nonlocal operator

In this section we describe our higher order integration method for the convolution;

$$\int_0^{\infty}\frac{k^+(s)+k^-(s)}{2}(u(x+s)-2u(x)+u(x-s))\,ds+\int_0^{\infty}\frac{k^+(s)-k^-(s)}{2}(u(x+s)-u(x-s))\,ds.$$

For the symmetric part,

$$\int_{-\infty}^{\infty} s^2k_s(s)\,\frac{u(x+s)-2u(x)+u(x-s)}{s^2}\,ds,$$

where we have

$$\frac{u(x+s)-2u(x)+u(x-s)}{s^2}\sim u_{xx}(x)+\frac{s^2}{12}u_{xxxx}(x)+O(s^4)$$

We apply the fourth order approximation of u_{xx} by

$$u_{xx}(x)\sim\frac{u(x+h)-2u(x)+u(x-h)}{h^2}-\frac{1}{12}\frac{u(x+2h)-4u(x)+6u(x)-4u(x-h)+u(x-2h)}{h^2}$$

and we apply the second order approximation of $u_{xxxx}(x)$ by

$$u_{xxxx}(x) \sim \frac{u(x+2h)-4u(x)+6u(x)-4u(x-h)+u(x-2h)}{h^4}.$$

Thus, one can approximate

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} s^2 k_s(s)\, \frac{u(x+s)-2u(x)+u(x-s)}{s^2}\, ds$$

by

$$\rho_0\,(\frac{u_{k+1}-2u_k+u_{k-1}}{h^2} - \frac{1}{12}\frac{u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}}{h^2})$$
$$+\frac{\rho_1}{12}\,\frac{u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}}{h^2},$$

where

$$\rho_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} s^2 k_s(s)\, ds \quad \text{and} \quad \rho_1 = \frac{1}{h^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} s^4 k_s(s)\, ds.$$

The remaining part of the convolution

$$\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} u(x_{k+j}+s) k_s(s)\, ds$$

can be approximated by three point quadrature rule based on

$$u(x_{k+j}+s) \sim u(x_{k+j}) + u'(x_{k+j})s + \frac{s^2}{2}u''(x_{k+j})$$

with

$$u'(x_{k+j}) \sim \frac{u_{k+j+1}-u_{k+j-1}}{2h}$$

$$u''(x_{k+j}) \sim \frac{u_{k+j+1}-2u_{k+j}+u_{k+j-1}}{h^2}.$$

That is,

$$\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} u(x_{k+j}+s) k_s(s)\, ds$$

$$\sim \rho_0^k u_{k+j} + \rho_1^k \frac{u_{k+j-1}-u_{k+j+1}}{2} + \rho_2^k \frac{u_{j+k+1}-2u_{k+j}+u_{j+k-1}}{2}$$

where

$$\rho_0^k = \textstyle\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} k_s(s)\, ds$$

$$\rho_1^k = \tfrac{1}{h}\textstyle\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} (s-x_k) k_s(s)\, ds$$

$$\rho_2^k = \tfrac{1}{h^2}\textstyle\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} (s-x_k)^2 k_s(s)\, ds.$$

For the skew-symmetric integral

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} k_u(s)(u(x+s)-u(x-s))\,ds \sim \rho_2\, u_x(x) + \frac{\rho_3}{6} h^2\, u_{xxx}(x)$$

where

$$\rho_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} 2sk_u(s)\,ds, \quad \rho_3 = \frac{1}{h^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} 2s^3k_u(s)\,ds.$$

We may use the forth order difference approximation

$$u_x(x) \sim \frac{u(x+h)-u(x-h)}{2h} - \frac{u(x+2h)-2u(x+h)+2u(x-h)-u(x-2h)}{6h}$$

and the second order difference approximation

$$u_{xxx}(x) \sim \frac{u(x+2h)-2u(x+h)+2u(x-h)-u(x-2h)}{h^3}$$

and obtain

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} k_u(s)(u(x+s)-u(x-s))\,ds$$

$$\sim \rho_2\,(\frac{u_{k+1}-u_{k-1}}{2h} - \frac{u_{k+2}-2u_{k+1}+2u_{k-1}-u_{k-1}}{6h}) + \frac{\rho_3}{6}\frac{u_{k+2}-2u_{k+1}+2u_{k-1}-u_{k-1}}{h}.$$

15 Eigenvalue Problems for Fractional Operators

In this section we consider the eigenvalue problem for the fractional differential operator. Given the potential function $q \in L^\infty(0,1)$ consider the eigenvalue problem

$$\mathcal{A}u = -\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} u''(s)\,ds + q(t)u(t) = \lambda\, u(t)$$

with

$$dom\,(A) = \{u \in H^1(0,1) : u' \in dom(D_t^\alpha) \text{ with } u(0)=u(1)=0\}.$$

Since

$$\int_0^1 (\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} u''(s)\,ds)\phi(t)\,dt = \int_0^1 (\int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\phi(t)\,dt)u''(s)\,ds$$

$$= u'(1)(\int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\phi(t)\,dt)(1^-) - u'(0)\int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\phi(t)\,dt - \int_0^1 u'(s)\frac{d}{ds}\int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\phi(t)\,dt$$

$$= u'(1)(\int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\phi(t)\,dt)(1^-) - u'(0)\int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\phi(t)\,dt + \int_0^1 u(s)\frac{d^2}{ds^2}\int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\phi(t)\,dt\,ds,$$

the adjoint operator of $\mathcal{A}$ is given by

$$\mathcal{A}^*\phi = -\frac{d}{ds}\int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\phi'(t)\,dt + q(s)\phi(s)$$

with

$$dom\,(\mathcal{A}^*) = \{\phi \in L^2(0,1) : \int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\phi(t)\,dt \in H^2(0,1)\cap H_0^1(0,1)\}$$

$$= \{\phi \in H^1(0,1) : \phi(1)=0;\quad \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\phi(t)\,dt = 0,\quad \int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\phi'(t)\,dt \in H^1(0,1)\}$$

Thus, $dom\,(\mathcal{A}) \neq dom\,(\mathcal{A}^*)$ and $\mathcal{A}$ is not self-adjoint. Note that

$$(\phi, A^*\phi) = \int_0^1 \phi'(s) D_s^1 \phi\,ds.$$

With zero extension of ϕ to we have

$$\int_0^1 \phi'(s) D_s^1 \phi\,ds. = \int (i\,\omega)^{1+\alpha} |\hat{\phi}(\omega)|^2$$

where $\hat{\phi}$ is the Fourier transform of ϕ.

We develop a numerical method that approximates $\mathcal{A}$ and $\mathcal{A}^*$ simultaneously. It is based on the Legendre-tau method []. We use the Legendre approximation

$$\tilde{u}^N(t) = \sum_{k=0}^N u_k\, L_k(2t-1)$$

where $L_k(\cdot)$ is the k-th Legendre polynomial on $[-1,1]$. The boundary condition $u^N(0) = u^N(1) = 0$ implies

$$u_N = -\sum_{k:even} u_k, \quad u_{N-1} = \sum_{k:odd} u_k \tag{15.1}$$

Leg

if N is even and

$$u_N = -\sum_{k:odd} u_k, \quad u_{N-1} = \sum_{k:even} u_k \tag{15.2}$$

Leg2

if N is odd. Thus, $\{u_k\}_{k=0}^{N-2}$ defines the approximations

$$u^N = \sum_{k=0}^{N-2} u_k\, L_k(2t-1)$$

and

$$\tilde{u}^N = u^N + u_{N-1}\, L_{N-1}(2t-1) + u_N\, L_N(2t-1).$$

where u_{N-1} and u_N are defined by (15.1)–(15.2). The Legendre-tau approximation $\mathcal{A}^N : X^{n-2} \to X^{N-2}$ is defined by

$$\mathcal{A}^N u^N = P^{N-2}\mathcal{A}\tilde{u}^N,$$

where P^{N-2} is the orthogonal projection of $L^2(0,1)$ onto

$$X^{N-2} = \{u \in L^2(0,1); u = \sum_{k=0}^{N-2} u_k\, L_k(2t-1)\}.$$

That is,

$$(\mathcal{A}^N u^N)_j = \int_0^1 L_j(2t-1)\left(-\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}(\tilde{u}^N)'(s)\,ds + q(t)u^N(t)\right)\,dt.$$

Since $(\tilde{u}^N)'' \in X^{N-2}$, for $\phi^N \in X^{N-2}$

$$(\mathcal{A}^N u^N, \phi^N) = \int_0^1 (\tilde{u}^N)''(P^{N-2}\int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\phi^N(t)\,dt + \beta_{N-1}L_{N-1}(2s-1) + \beta_N L_N(2s-1))$$

where we let

$$\beta_N = -\sum_{k:even} \beta_k, \quad \beta_{N-1} = -\sum_{k:odd} \beta_k$$

and

$$P^{N-2}\left(\int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\phi^N(t)\,dt\right) = \sum_{k=0}^{N-2} \beta_k\, L_k(2s-1).$$

It thus follows that $(\mathcal{A}^*)^N \in \mathcal{L}(X^{N-2}, X^{N-2})$ is given by

$$(\mathcal{A}^*)^N \phi^N = \frac{d^2}{ds^2}\left(P^{N-2}\int_s^1 \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\phi^N(t)\,dt + \beta_{N-1}L_{N-1}(2s-1) + \beta_N L_N(2s-1)\right) + P^{N-2}(q(s)\phi^N).$$

16 Appendix: Derivation

Let us consider the continuous time random walk (CTRW) process $x(t)$ on R which is characterized by the joint probability density function (pdf) $\varphi(\xi, \tau)$ of jumps $\xi = x(t_j) - x(t_{j-1})$ and waiting times $\tau_j = t_j - t_{j-1}$. We assume jumps and waiting times are independent, i.e., $\varphi = \lambda(\xi)\psi(\tau)$. The jump pdf $\lambda(\xi)$ represents the pdf of jump size ξ and the waiting time pdf $\psi(\tau)$ represents the pdf of waiting time τ. Thus,

$$\int_0^t \psi(\tau)\,d\tau$$

gives the probability tat at least one jump is taken in $(0, t)$.

$$\Psi(t) = 1 - \int_0^t \psi(\tau)\,d\tau$$

defines the probability of no jump occurs during $(0, t]$. Let us denote by $p(x, t)$ the pdf of reaching to position x after time t, i.e. $p(x, 0) = \delta(x)$. The master equation of CTRW is given by

$$p(x,t) = \delta(x)\Psi(t) + \int_0^t \psi(t-t')\int_{-\infty}^{\infty} \lambda(x-x')p(x',t')\,dx'dt'. \tag{16.1}$$ CTRW

Taking the Laplace transform in t and the Fourier transform in x of (CTRW 16.1), we obtain the Montroll-Weiss equation

$$\hat{p}(k,s) = \frac{\hat{\Psi}(s)}{1-\hat{\lambda}(k)\hat{\psi}(s)} = \frac{1-\hat{\psi}(s)}{s}\frac{1}{1-\hat{\lambda}(k)\hat{\psi}(s)}. \tag{16.2}$$ MW

In order to derive an evolution equation of Fokker-Planck-Kolmogorov type we rewrite (Mw 16.2) as

$$\hat{\Phi}(s)(s\hat{p}(k,s)-1) = (\hat{\lambda}(k)-1)\hat{p}(k,s), \tag{16.3}$$ CTRW1

where

$$\hat{\Phi}(s) = \frac{1-\hat{\psi}(s)}{s\hat{\psi}(s)} = \frac{\hat{\Psi}(s)}{\hat{\psi}(s)} = \frac{\hat{\Psi}(s)}{1-s\hat{\Psi}(s)}.$$

Taking the inverse transforms of (CTRW1 16.3), we obtain the evolution equation for p;

$$\int_0^t \Phi(t-s)\frac{\partial}{\partial t}p(x,s)\,ds = -p(x,t) + \int_{-\infty}^{\infty} \lambda(x-x')p(x',t)\,dx', \tag{16.4}$$ CTRW2

where

$$\Phi(t) = \mathcal{L}^{-1}\{\frac{\hat{\Psi}(s)}{1-s\hat{\Psi}(s)}\}$$

and

$$\Psi(t) = \int_0^t \Phi(t-s)\psi(s)\,ds.$$

If $\hat{\Phi}(s)=1$ and thus $\psi(t)=\Psi(t)=e^{-t}$, it reduces to the Kolmogorov-Feller equation

$$\frac{\partial}{\partial t}p(x,t) = -p(x,t) + \int_{-\infty}^{\infty} \lambda(x-x')p(x',t)\,dx'.$$

If we assume λ is symmetric,

$$-p(x,t) + \int_{-\infty}^{\infty} \lambda(x-x')p(x',t)\,dx' = \int_0^{\infty} \lambda(s)(p(x+s)-2p(x)+p(x-s))\,ds$$

and thus

$$\int_{-\infty}^{\infty}[\int_0^{\infty} \lambda(s)(p(x+s)-2p(x)+p(x-s))\,ds\psi(x)\,dx$$

$$= -\int_0^{\infty} \lambda(s)\int_{-\infty}^{\infty}(p(x+s)-p(x))(\psi(x+s)-\psi(x))\,dx.$$

Hence, the right hand side defines a self-adjoint nonnegative definite operator A on $L^2(R)$.

17 Fractional diffusion equation via Homogenization

In this section we discuss an example of fractional diffusion equation which is derived by the homogenization method. is presented. This method uses a small parameter which measures the characteristic length of the period (i.e. of the heterogeneities) compared to a macroscopic length. In Section 3, the classical

Consider a diffusive mass transport of chemical specifies through a rigid porous saturated composite, consisting of tow porous materials;

$$\frac{\partial c}{\partial t} = \nabla \cdot (D(\frac{x}{\epsilon})\nabla c + b(\frac{x}{\epsilon})\, c) \tag{17.1}$$ model

where we assume the periodic diffusive media $D(\frac{x}{\epsilon})$ and the periodic advection $b(\frac{x}{\epsilon})$ with period ϵ and they are given by

$$D(y) = \epsilon^2 D_2\, \chi_{\Omega_2}(y) + D_1\, \chi_{\Omega_1}(y), \quad b(y) = \epsilon^2 b_2\, \chi_{\Omega_2}(y) + b_1\, \chi_{\Omega_1}(y),$$

Here subdomains Ω_1 and Ω_2 are for each composite and are disjoint and

$$\overline{\Omega_1} \cup \overline{\Omega_2} = [0,1]^d = \Omega. \quad \Gamma = \overline{\Omega_1} \cap \overline{\Omega_2}.$$

Here $\epsilon > 0$ is a small parameter which measures the characteristic length of the period of the heterogeneities compared to a macroscopic length. the heterogeneity reflects that Ω_1 is a diffusive (fluid) medium and Ω_2 is a less diffusive (solid) medium with ration ϵ^2. It follows from [Auriault&Lewandowka] that the homogenized equation as $\epsilon \to^+$ is given by

$$\frac{|\Omega_1|}{|\Omega|}\frac{\partial c}{\partial t} + \int_0^t K(t-s)\frac{\partial c}{\partial s}\, ds = \nabla \cdot (D_{\text{eff}} \nabla c + b_{\text{eff}}\, c), \tag{17.2}$$ hom0

where

$$\mathcal{L}(K) = \langle 1 - \hat{k} \rangle = \frac{1}{|\Omega|}\int_{\Omega_2} (1 - \hat{k})\, dy$$

and the $Y-$periodic $\hat{k}$ satisfies

$$\nabla_y \cdot (D_2 \nabla_y \hat{k}) = p\,(\hat{k} - 1), \quad y \in \Omega_2 \quad \hat{k} = 0 \text{ at } \Gamma.$$

and D_{eff} and b_{eff} are defined by (homo3 17.11).

In what follows we give a sketch of the derivation of (hom0 17.2). Let $y = \frac{x}{\epsilon}$ and assume the expansion;

$$c_1 = c_1^0(x,y,t) + \epsilon\, c_1^1(x,y,t) + \epsilon^2\, c_1^2(x,y,t) + \cdots$$

$$c_2 = c_2^0(x,y,t) + \epsilon\, c_2^1(x,y,t) + \epsilon^2\, c_2^2(x,y,t) + \cdots.$$

Substituting this into (model 17.1) and using the calculus;

$$\nabla c(x,y,t) = \nabla_x c(x,y,t) + \frac{1}{\epsilon}\nabla_y c(x,y,t)$$

and taking the Laplace transform of the resulting equation in time, it results in the following order terms.

ϵ^{-2} order:

$$\nabla_y \cdot (D_1 \nabla_y \hat{c}_1^0) = 0, \quad y \in \Omega_1 \quad \text{and} \quad n \cdot (D_1 \nabla_y \hat{c}_1^0) = 0, \quad y \in \Gamma \tag{17.3}$$ order-2

ϵ^{-1} order:

$$\begin{aligned} &\nabla_y \cdot (D_1 \nabla_x \hat{c}_1^0 + b_1 \hat{c}_1^0) + \nabla_x \cdot (D_1 \nabla_y \hat{c}_1^0) + \nabla_y \cdot (D_1 \nabla_x \hat{c}_1^1) = 0, \quad y \in \Omega_1 \\ &n \cdot (D_1 \nabla_y \hat{c}_1^1 + D_1 \nabla_x \hat{c}_1^0 + b_1 \hat{c}_1^0) = 0, \quad y \in \Gamma \end{aligned} \tag{17.4}$$ order-1

ϵ^0 order:

$$\nabla_x \cdot (D_1 \nabla_x \hat{c}_1^0 + b_1 \hat{c}_1^0) + \nabla_x \cdot (D_1 \nabla_y \hat{c}_1^1) + \nabla_y \cdot (D_1 \nabla_x \hat{c}_1^1 + b_1 \hat{c}_1^1) + \nabla_y \cdot (D_1 \nabla_y \hat{c}_1^2) = p\hat{c}_1^0, \ \ y \in \Omega_1$$

$$n \cdot (D_1 \nabla_y \hat{c}_1^2 + D \nabla_x \hat{c}_1^1 + b_1 \hat{c}_1^1) = n \cdot (D_2 \nabla_y \hat{c}_2^0), \ \ y \in \Gamma \tag{17.5}$$

order0-1

and

$$\nabla_y \cdot (D_2 \nabla_y \hat{c}_2^0) = p\hat{c}_2^0, \ \ y \in \Omega_2. \tag{17.6}$$

order0-2

From (17.3) $\hat{c}_1^0 = \hat{c}(x,p)$ in which (x,p) acts as the parameter. Thus, from (17.4) we have

$$\nabla_y \cdot (D_1 \nabla_x \hat{c}_1^0 + b_1 \hat{c}_1^0) + + \nabla_y \cdot (D_1 \nabla_y \hat{c}_1^1) = 0, \ \ y \in \Omega_1$$

$$n \cdot (D_1 \nabla_y \hat{c}_1^1 + D_1 \nabla_x \hat{c}_1^0 + b_1 \hat{c}_1^0) = 0, \ \ y \in \Gamma$$

and for $\psi \in H^1_{\text{per}}(Y)$ we have

$$\int_{\Omega_1} (D_1 \nabla_y \hat{c}_1^1, \nabla_y \psi) = -\sum_{i,j} \frac{\partial}{\partial x_j} \hat{c}_0^1 \int_{\Omega_1} (D_1)_{ij} \frac{\partial \psi}{\partial y_i} - \hat{c}_1^0 \int_{\Omega_1} b_1, \nabla_y \psi)\, dy \tag{17.7}$$

id1

Let $Y-$ periodic functions $w^k, \ v \in\in H^1(\Omega_1)$ satisfy

$$\int_{\Omega_1} (D_1 \nabla_y w^k, \nabla_y \psi) = -\int_{\Omega_1} (D_1 e^k, \nabla_y \psi)\, dy$$

$$\int_{\Omega_1} (D_1 \nabla_y v, \nabla_y \psi) = -\int_{\Omega_1} (b_1, \nabla_y \psi)\, dy \tag{17.8}$$

homo

for all $\psi \in H^1_{\text{per}}(Y)$. It follows from (17.7) that

$$\hat{c}_1^1 = \sum_k w^k(y) \frac{\partial}{\partial x_k} \hat{c}_1^0 + v(y)\, \hat{c}_1^0 + \tilde{c}(x,p). \tag{17.9}$$

homo1

From (17.6)

$$\nabla_y \cdot (D_2 \nabla_y (\hat{c}_2^0 - \hat{c}) = p(\hat{c}_2^0 - \hat{c} + \hat{c}), \quad \hat{c}_2^0 - \hat{c} = 0 \text{ at } \Gamma,$$

and thus

$$\hat{c}_2^0 = (1 - \hat{k})\hat{c}, \tag{17.10}$$

homo2

where $Y-$periodic function $\hat{k}$ satisfies

$$\nabla_y \cdot (D_2 \nabla_y \hat{k}) = p(\hat{k} - 1), \ \ y \in \Omega_2, \quad \hat{k} = 0 \text{ on } \Gamma.$$

Integrating (17.5) with respect to y on Ω_1, we obtain

$$\int_{\Omega_1} \nabla_x \cdot (D_1 \nabla_x \hat{c}_1^0 + b_1 \hat{c}_1^0)\, dy + \int_{\Omega_1} \nabla_x \cdot (D_1 \nabla_y \hat{c}_1^1)\, dy + \int_\Gamma n \cdot (D_2 \hat{c}_2^0)\, ds = p|\Omega_1|\, \hat{c}$$

From (17.9)

$$\frac{1}{|\Omega|}\Big(\int_{\Omega_1} \nabla_x \cdot (D_1 \nabla_x \hat{c}_1^0 + b_1 \hat{c}_1^0)\, dy + \int_{\Omega_1} \nabla_x \cdot (D_1 \nabla_y \hat{c}_1^1)\, dy\Big) = \nabla_x (D_{\text{eff}} \nabla_x \hat{c} + b_{\text{eff}} \hat{c})$$

where

$$(D_{\text{eff}})_{kj} = \frac{1}{|\Omega|}\int_{\Omega_1} D_1(I_{kj} + \frac{\partial w^k}{\partial y_j})\,dy$$

$$b_{\text{eff}} = \frac{1}{|\Omega|}\int_{\Omega_1}(b_1 + D_1\nabla v)\,dy. \tag{17.11}$$

homo3

From (17.6) and (17.10)

$$\frac{1}{|\Omega|}\int_{\Gamma} n\cdot(D_2\hat{c}_2^0)\,ds = -\frac{1}{|\Omega|}\int_{\Omega_1} p(1-\hat{k})\,dy\,\hat{c}$$

Hence we obtain

$$\nabla_x(D_{\text{eff}}\nabla_x\hat{c} + b_{\text{eff}}\hat{c}) + \frac{1}{|\Omega|}\int_{\Omega_1}(1-\hat{k})\,dy\,p\hat{c} = \frac{|\Omega_1|}{|\Omega|}p\hat{c}.$$

which implies (17.2).

If $\hat{u} = 1-\hat{k}$ we have

$$D_2\,\Delta\hat{u} = p\,\hat{u}$$

$$\hat{u} = 1 \quad \text{at } \Gamma \tag{17.12}$$

hom

For $\Omega_2 = \{|r| \le r_0\}$ in R^3 the radial solution $\hat{u} = \hat{u}(r)$ satisfies

$$(ru)'' = pr\,u(r)$$

Thus,

$$\hat{u}(r) = \frac{r_0}{r}\frac{\sinh(\sqrt{p}r)}{\sinh(\sqrt{p}r_0)}$$

and

$$\langle\hat{u}\rangle = \frac{4\pi r_0^2}{\sqrt{p}}\coth(\sqrt{p}r_0) - \frac{4\pi r_0}{p}.$$

In general, from (17.12) by the divergence theory

$$p\langle\hat{u}\rangle = \int_{\Gamma}\frac{\partial}{\partial\nu}\hat{u}\,ds.$$

Since at $x^* \in \Gamma$

$$\frac{\partial^2}{\partial\nu^2}\hat{u} + \Delta_\tau\hat{u} = p\hat{u}$$

and $\hat{u} = 1$ on Γ, we have

$$\frac{\partial^2}{\partial\nu^2}\hat{u} - \kappa\,\frac{\partial}{\partial\nu}\hat{u} = p\hat{u},$$

where κ is the curvature of Γ at x^*. It can be shown that

$$\frac{\partial}{\partial\nu}\hat{u} \sim \sqrt{p}$$

for $p >> 1$, since

$$\hat{u} \sim e^{\sqrt{p}\,(x-x^*)\cdot\nu} \text{ at } x^* \in \Gamma.$$

Thus, for $p >> 1$

$$\langle \hat{u} \rangle \sim \frac{|\Gamma|}{|\Omega|} \frac{1}{\sqrt{p}}$$

and

$$K(t) \sim \frac{|\Gamma|}{|\Omega|} \frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}.$$

It thus follows from (hom0 17.2) that

$$\frac{|\Omega_1|}{|\Omega|} \frac{\partial}{\partial t} c + \frac{|\Gamma|}{|\Omega|} D_t^{\frac{1}{2}} c = \nabla \cdot (D_{\text{eff}} \nabla c + b_{\text{eff}}\, c).$$

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