
Feedback and Time Optimal Control for Quantum Spin Systems

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— Joint Work with Karl Kunisch, K.F. University of Garz and Qin Zhang.

1. MODEL PROBLEMS

Abstract Schrödinger Control System

$$i \frac{\partial}{\partial t} \Psi(x, t) = (\mathcal{H}_0 + \epsilon(t)\mu) \Psi(x, t) + \gamma |\Psi|^2 \Psi, \quad \Psi(x, 0) = \Psi_0(x).$$

Let the internal Hamiltonian \mathcal{H}_0 is a positive closed self-adjoint operator on a Hilbert space X . In the presence of an external interaction taken as an electric field modelled by a coupling operator with amplitude $\epsilon(t) \in R$ and a time independent dipole moment operator μ , the new Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \sum_{j=1}^m \epsilon_j(t) \mu_j$ gives rise to the dynamical equations to be controlled, (e.g, $\mu_j \Psi = v_j(x) \Psi$)

Control Problem We consider the control problem of driving the state $\Psi(t)$ to an energy equilibrium state of $\bar{\Psi}$ of \mathcal{H}_0 , i.e. $\bar{\Psi} = e^{i\phi} \bar{\psi}$ where $\mathcal{H}_0 \bar{\psi} = \lambda \bar{\psi}$.

Linear Schrödinger Equations: $\gamma = 0$

$$\frac{\partial}{\partial t} \Psi_1(x, t) = (\mathcal{H}_0 + \epsilon(t)\mu) \Psi_2(x, t) + \gamma |\Psi|^2 \Psi_2$$

$$\frac{\partial}{\partial t} \Psi_2(x, t) = -(\mathcal{H}_0 + \epsilon(t)\mu) \Psi_1(x, t) - \gamma |\Psi|^2 \Psi_1,$$

i.e., for $\Psi(x, t) = \Psi_1(x, t) + i \Psi_2(x, t)$ and $\Psi = (\Psi_1, \Psi_2)$,

$$\frac{d}{dt} \Psi = A\Psi + \epsilon(t)B\Psi \quad \text{with}$$

$$A = \begin{pmatrix} 0 & \mathcal{H}_0 \\ -\mathcal{H}_0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}$$

Multiple controls: $\mu(t) = \sum_{j=1}^m \epsilon_j(t) \mu_j$.

$$\text{Mild Solution: } \Psi(t) = S(t)\Psi_0 + \int_0^t S(t-s)\epsilon(s)B\Psi(s) ds.$$

Orbit Tracking:

$$V = V(\Psi(t), \mathcal{O}(t)) = \frac{1}{2} |\Psi(t) - \mathcal{O}(t)|_X^2 = 1 - \operatorname{Re}(\mathcal{O}, \Psi)_H$$
$$i \frac{\partial}{\partial t} \mathcal{O}(t) = \mathcal{H}_0 \mathcal{O}(t), \text{ e.g. } \mathcal{O}(t) = e^{-i(\lambda t - \theta)} \psi, \quad H = \lambda \psi$$

$$\frac{d}{dt} V(\Psi(t), \mathcal{O}(t)) = \epsilon(t) \operatorname{Im}(\mathcal{O}(t), \mu \Psi(t))$$

$$\epsilon(t) = -\frac{1}{\alpha} (u(t) + \beta \operatorname{sign}(u(t)) V^\gamma), \quad u(t) = \operatorname{Im}(\mathcal{O}(t), \mu \Psi(t))$$

$$\frac{d}{dt} V(\Psi(t), \mathcal{O}(t)) = -\frac{1}{\alpha} (|u(t)|^2 + \beta |u(t)| V^\gamma).$$

[K. Beauchard, J.M. Coron, M. Mirrahimi, and P. Rouchon] Implicit Lyapunov control of finite dimensional Schrödinger equations, preprint.

[M. Mirrahimi, P. Rouchon, and G. Turinici] Lyapunov control of bilinear Schrödinger equations, *Automatica*, 41(2005), 1987-1994.

Optimality: $V = V(\Psi, \mathcal{O}(t))$ satisfies HJB equation

$$\frac{\partial}{\partial t} V(\Psi, \mathcal{O}(t)) + \min_{\epsilon} [(A_0 \Psi + \epsilon B \Psi, V_{\Psi}) + \frac{1}{\alpha} (|u(t)|^2 + \beta |u(t)| V^{\gamma}) + \alpha |\epsilon + \frac{\beta}{\alpha} \text{sign}(u(t)) V^{\gamma}|^2)] = 0,$$

$$u(t) = \text{Im}(\mathcal{O}(t), \mu \Psi)$$

Asymptotic Tracking $V(\Psi, \mathcal{O}) \rightarrow 0$ as $t \rightarrow \infty$ Since

$$V(\Psi(t), \mathcal{O}(t)) + \int_0^t (\beta |u(s)|^2 + |u(s)| V(\Psi(s), \mathcal{O}(s))) ds = V(\Psi(0), \mathcal{O}(0)).$$

$V(\Psi(t), \mathcal{O}(t))$ is monotonically decreasing in t . Using the compactness of the orbit we prove the following invariance principle.

Invariance Principle Suppose

$$\int_0^T \alpha |\operatorname{Im}(\mathcal{O}_\infty(t), \mu \Psi_\infty(t))|^2 dt = 0.$$

$$i \frac{d}{dt} \Psi_\infty = \mathcal{H}_0 \Psi_\infty, \quad \Psi_\infty(t) = \omega - \text{Limit} = \sum_{k=1}^{\infty} A_k e^{i(-\lambda_k t + \theta_k)} \psi_k.$$

where for $\mu_{k_0}^k = (\psi_{k_0}, \mu \psi_k)_H$ and $\mathcal{O}_\infty(\tau) = e^{i(\lambda_{k_0} \tau - \tilde{\theta}_{k_0})} \psi_{k_0}$,

$$\operatorname{Im}(\mathcal{O}(t), \mu \Psi(t)) = \operatorname{Im}\left(\sum_{k=1}^{\infty} A_k e^{i((\lambda_k - \lambda_{k_0})\tau - \theta_k + \tilde{\theta}_{k_0})} \mu_{k_0}^k\right).$$

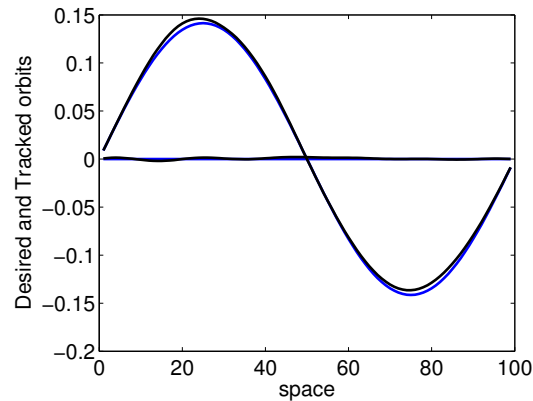
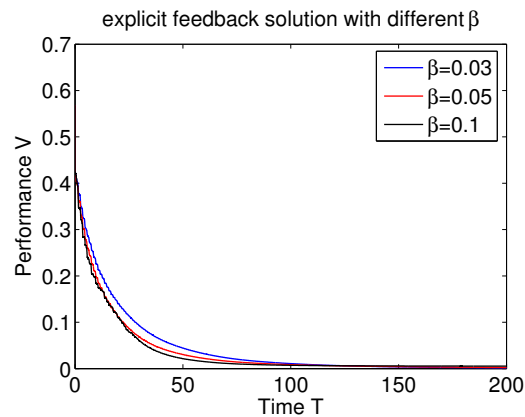
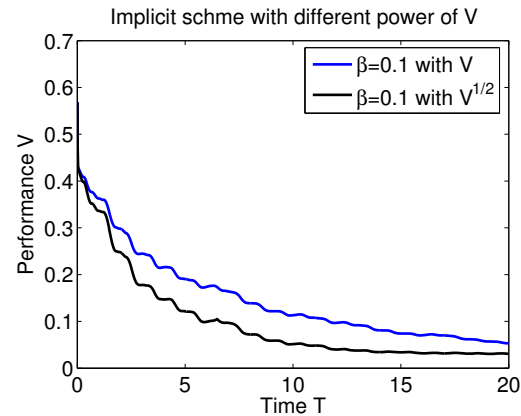
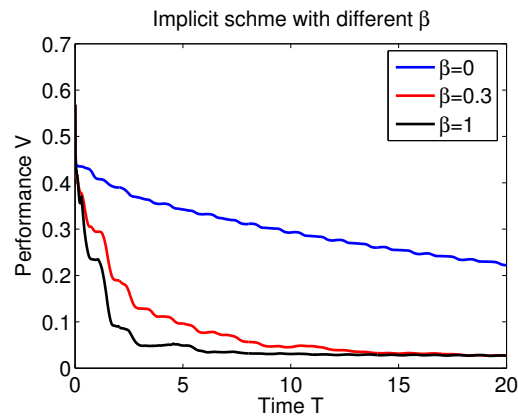
Ingham's Lemma Assume $|\mu_m^{k=1} - \mu_\ell| \geq \delta$, $m \neq \ell$ for

$$\mu_k = \lambda_k - \lambda_{k_0}, \quad k \geq 1, \quad \mu_{-k} = -(\lambda_k - \lambda_{k_0}) \quad k \neq k_0.$$

If $T > \frac{2\pi}{\delta}$, there exists a constant c , depending on T and $\delta > 0$ such that

$$c \sum_{m \in I} |a_m|^2 \leq \int_0^T |f(\tau)|^2 d\tau, \quad f(\tau) = \sum_{m \in I} a_m e^{i\mu_m \tau}.$$

Thus, $A_k = 0$, $k \neq k_0$ and $\tilde{\theta}_{k_0} = \theta_{k_0}$ if $\mu_{k_0}^k \neq 0$.



$\beta = 1/500$, $\bar{\Psi} = \sin(2\pi x)$, $\Psi_0 = e^{-100(x-.8)^2}$, $\mu_i \Psi = v_i(x) \Psi(x)$ with
 $v_1(x) = x - 0.5 - 1.75(x - 0.5)^2$, and $v_2(x) = x - 0.5 - 1.75(x - 0.5)^2 + 2.5(x - 0.5)^3 - 2.5(x - 0.5)^4$.

Feedback Control of Quantum Spin Systems

Observable process: $X_t = U_t^* X U_t$, where the unitary operator satisfies the Schrodinger type quantum stochastic differential equation (QSDE)

$$dU_t = \left(-iHdt - \frac{1}{2}L^*Ldt + LdA_t^* - L^*dA_t\right)U_t, \quad U_0 = I \quad (1)$$

where A_t is the annihilation and A_t^* is the creation operator and L is an atomic operator. The homodyne detection process:

$$dY_t = U_t^*(L + L^*)U_t^+ dA_t + dA_t^*$$

By using Quantum Ito's rule, we get the QSDE for $X(t)$

$$dX_t = j_t(\mathcal{L}_{L,H}(X))dt + j_t([L^*, X])dA_t + j_t([X, L])dA_t^*,$$

where $j_t(X) = U_t^* X U_t$ and the (Lindblad) generator given by

$$\mathcal{L}_{L,H}(X) = i[H, X] + L^*XL - \frac{1}{2}(XL^*L + L^*LX).$$

K. R. Parthasarathy, An Introduction to Quantum Stochastic Calculus, Birkhauser, Basel, 1992

Luc. Bouten, Ramon van Handel, Matthew R James: An Introduction to Quantum Filtering, arXiv:math. OC/-601741v1, 2006.

M. Mirrahimi, R.A.Handel, Stabilizing Feedback Controls for Quantum Systems, arXiv:math.ph/0510066v2, 2005.

K. Tsumura, Global Stabilization of N-dimensional Quantum Spin Systems via Continuous Feedback.

Follow the quantum filtering theory, the conditional expectation of the observable X , $\mathbb{P}(X_t|\mathcal{Y}_t) = \pi_t(X) = \text{Tr}(\rho_t X)$ is given by

$$d\rho_t = (-i[H, \rho_t] + L\rho_t L^* - \frac{1}{2}\rho_t L^* L - \frac{1}{2}L^* L\rho_t)dt \\ + (L\rho_t + \rho_t L^* - \text{Tr}[\rho_t(L + L^*)]\rho_t)dW_t$$

where The Brownian process $dW_t = dY_t - \text{Tr}[(L + L^*)\rho_t]dt$ is the innovation process.

Consider a quantum spin system with fixed angular momentum J . Let $H_t = u_t F_y$ be the control Hamiltonian and $L = F_z$ be the coupling operator. Then the corresponding quantum filtering equation is

$$d\rho_t = -iu_t[F_y, \rho_t]dt - \frac{1}{2}[F_z, [F_z, \rho_t]]dt \\ + \sqrt{\eta}(F_z\rho_t + \rho_t F_z - 2\text{Tr}[F_z\rho_t]\rho_t)dW_t$$

where F_y and F_z are self adjoint angular momentum operators along axis y and z .

$$F_y = \frac{1}{2i} \begin{pmatrix} 0 & -c_1 & & & \\ c_1 & 0 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{2J-1} & 0 & -c_{2J} \\ & & & c_{2J} & 0 \end{pmatrix}, c_m = \sqrt{(2J+1-m)m}$$

$$F_z = \begin{pmatrix} J & & & & \\ & J-1 & & & \\ & & \ddots & & \\ & & & -J+1 & \\ & & & & -J \end{pmatrix}$$

For any $\epsilon > 0$, let

$$\rho_{\tilde{f}} = (1 - \epsilon)\rho_f + \frac{\epsilon}{2J} \sum_{m \neq f} \rho_{\Psi_m}$$

where ρ_f , $f \in [-J, J]$ is an equilibrium solution ($\frac{d\rho_t}{dt}|_{\rho_t=\rho_f} = 0$). For $\alpha, \beta > 0$ consider the control law:

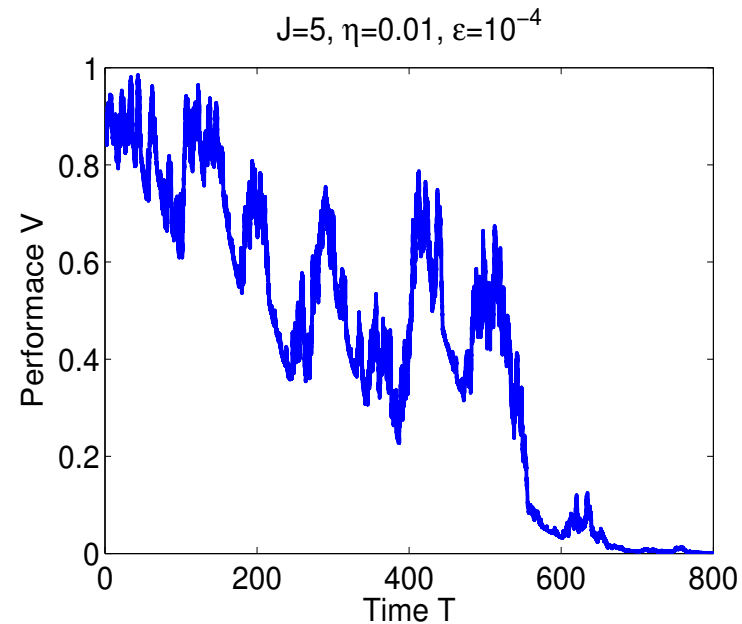
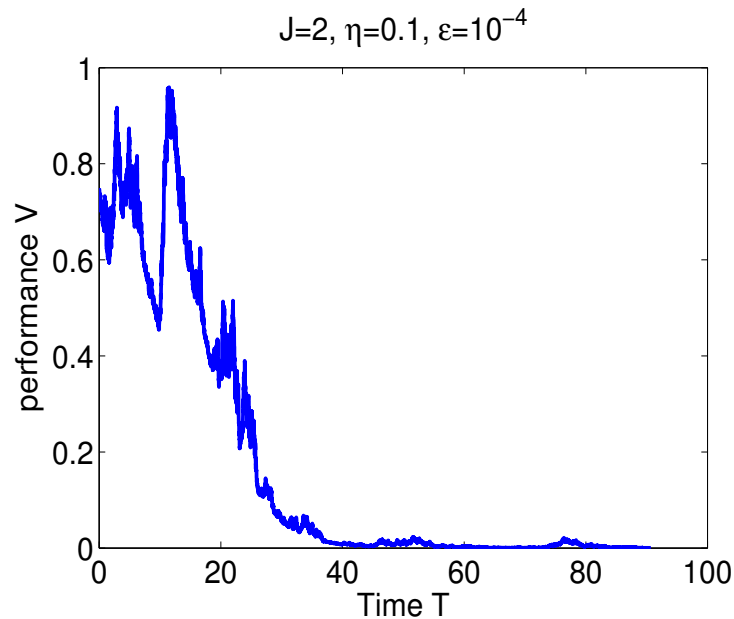
$$u_t = -(\alpha \tilde{u}_t + \beta \text{sign}(\tilde{u}_t) V_{\rho_f}^I(\rho_t)), \quad \tilde{u}_t = \text{Tr}(i[F_y, \rho_t] \rho_{\tilde{f}}).$$

Claim: $\rho_t \rightarrow \rho_f$ a.s. as $t \rightarrow \infty$. Let

$$V_{\rho_f}^I = 1 - \text{Tr}(\rho \rho_f), \quad V_{\rho_f}^{II} = 1 - (\text{Tr}(\rho \rho_f))^2$$

$$\mathcal{A}V^I = -u_t \text{Tr}(i[F_y, \rho_t] \rho_{\tilde{f}}) = -(\alpha |\tilde{u}_t|^2 + \beta |\tilde{u}_t| V^I(\rho_t))$$

$$\mathcal{A}V^{II} = -2\text{Tr}(\rho_t \rho_{\tilde{f}})(\alpha |\tilde{u}_t|^2 + \beta |\tilde{u}_t| V^I(\rho_t)) + 4\eta(f - \text{tr}(F_z \rho_t))^2 \text{tr}(\rho_t \rho_f)^2.$$



— Time Integration: Splitting of Deterministic and Stochastic terms.
Implicit Euler scheme is \mathcal{S} -invariant.

M. Mirrahimi, R.A.Handel, Stabilizing Feedback Controls for Quantum Systems, arXiv:math.ph/0510066v2, 2005.

Quantum Control System

$$\frac{d}{dt}\Psi(t) = -iH(t)\Psi(t)$$

with controlled Hamiltonian: $H(t) = H_0 + \sum_{i=1}^m \epsilon_i(t) H_i.$

Spin- $\frac{1}{2}$ system

$$\frac{d}{dt} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} 0 & I_z \\ -I_z & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} + \epsilon(t) \begin{pmatrix} 0 & I_x \\ -I_x & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

where the Pauli spin matrices are given by

$$I_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\Psi(0) = \frac{1}{\sqrt{2}} (1 \ 0 \ 0 \ 1), \quad \frac{1}{2} |\Psi(\tau) - \mathcal{O}|^2 \leq \delta, \quad \mathcal{O} = (1 \ 0 \ 0 \ 0).$$

Time Optimal control $x = (\Psi_1, \Psi_2)$. $u(t) = \epsilon(t)$

$$\min \int_0^\tau (1 + \epsilon |u(s)|^2) ds \quad \text{subject to}$$

$$\frac{d}{dt}x(t) = (A + u(t)B)x(t), \quad |u(t)| \leq \gamma \quad |x(\tau) - x_1|^2 \leq \delta, \quad x(0) = x_0 \in X.$$

Necessary optimality condition

$$\begin{aligned} \frac{d}{dt}x(t) &= f(x, u), \quad u(t) = -\text{sign}_\epsilon(\lambda(t) \cdot Bx(t)). \quad x(0) = x_0, \quad x(\tau) = x_1 \\ -\frac{d}{dt}\lambda(t) &= (A + uB)^t \lambda(t) \\ 1 + \lambda(\tau) \cdot (Ax(1) + u(1)Bx(1)) &= 0 \end{aligned}$$

$$\text{sign}_\epsilon(s) = \begin{cases} 1 & s \geq \epsilon \\ \frac{s}{\epsilon} & |s| \leq \epsilon \\ -1 & s \leq -\epsilon \end{cases}$$

With coordinate change $t = \tau s$, $F(z) = 0$ for $z = (x, \lambda, u, \tau) \in H^1(0, 1; R^n) \times H^1(0, 1, R^n) \times L^2(0, 1, R) \times R$:

$$\frac{d}{dt}x(t) = \tau (Ax + uBx)$$

$$u(t) = -\text{sign}_\epsilon(\lambda(t) \cdot Bx)$$

$$x(0) = x_0, \quad x(1) = x_1,$$

$$-\frac{d}{dt}\lambda(t) = \tau (A + uB)^t \lambda,$$

$$1 + \lambda(1) \cdot (Ax(1) + u(1)Bx(1)) = 0.$$

Semismooth Newton algorithm

$$z_{k+1} = z_k - J^{-1}F(z_k)$$

where $J\Delta z + F = 0$ is written as

$$\frac{d}{dt}\Delta x = \tau(A\Delta x + B\Delta u) + \Delta\tau(Ax + Bu) + f = 0 \quad \text{in } L^2(0, 1, R^n)$$

$$-\frac{d}{dt}\Delta\lambda = \tau A^* \Delta\lambda + \Delta\tau A^* \lambda + g = 0 \quad \text{in } L^2(0, 1, R^n)$$

$$\Delta u + G_\epsilon(B^* \lambda)B^* \Delta\lambda + e = 0 \quad \text{in } L^2(0, 1, R^n)$$

$$\Delta x(0) = 0, \quad \Delta x(1) + \Delta_1 = 0 \quad \text{in } R^n$$

$$\Delta\lambda(1) \cdot (Ax(1) + Bu(1)) + \lambda(1) \cdot (A\Delta x + B\Delta u) + \Delta_2 = 0 \text{ in } R.$$

$$\text{with } G_\epsilon(s) = \begin{cases} 0 & |s| \geq \epsilon \\ \frac{1}{\epsilon} & |s| < \epsilon \end{cases}$$

Semismooth Function: $F : D \subset X \rightarrow Z$ is called N -differentiable, if there exists a family of mappings $G : U \rightarrow \mathcal{L}(X, Z)$ such that

$$\lim_{|h| \rightarrow 0} \frac{|F(x+h) - F(x) - G(x+h)h|_Z}{|h|_X} = 0$$

for all $x \in U$. Moreover, F is semismooth at x if

$$\lim_{t \rightarrow 0^+} G(x + th)h \text{ exists uniformly in } |h| = 1.$$

Theorem (Local) Suppose F is semismooth at x^* and $|F'(x, h)| \geq \beta |h|$ for $\beta > 0$ and all $h \in X$. and assume for a N -derivative G in a neighborhood of x $|G(y)^{-1}| \leq 2\beta$ for all $y \in N(x^*)$. Then the Newton iterates:

$$x^{k+1} = x^k - G(x^k)^{-1} F(x^k)$$

are well-defined and converges to x^* superlinearly in a neighborhood $N(x^*)$.

$$|x^{k+1} - x^*| \leq |G(x^k)^{-1} (F(x^k) - F(x^*) - G(x^k)(x^k - x^*))| \leq o(|x^k - x^*|).$$

Solvability (Linear Case $\frac{d}{dt}x = Ax + Bu$): Let $(\Delta\lambda(1), \Delta\tau)$ be unknown.

$$\Delta\lambda(t) = e^{\tau A^*(1-t)} \Delta\lambda(1) + \int_t^1 e^{\tau A^*(s-t)} (A^* \lambda(s) \Delta\tau + g(s)) ds$$

$$\Delta x(t) = \int_0^t e^{\tau A(t-s)} (BG_\epsilon(b^* \lambda(s)) B^* \Delta\lambda(s) + (Ax + Bu) \Delta\tau + be(s) + g(s)) ds$$

Thus, we obtain the system of linear equations for $(\Delta\lambda(1), \Delta\tau)$:

$$\begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \begin{pmatrix} \Delta\lambda(1) \\ \Delta\tau \end{pmatrix} + q = 0 \quad (2)$$

where $q \in R^{n+1,1}$ depends on $F = (f, g, e, \Delta_1, \Delta_2)$ linearly. That is, if the matrix $\Phi \in R^{(n+1) \times (n+1)}$ is invertible, then J is bounded invertible.

Let $\{t : |B^* \lambda(t)| < \epsilon\} = \cup_i (t_i, t_{i+1})$ then

$$\Phi_{11} = \frac{1}{\epsilon} \sum_i e^{\tau A(1-t_i)} \mathcal{G}(\Delta t_i) e^{\tau A^*(1-t_i)}$$

$$\mathcal{G}(\Delta) = \int_0^\Delta e^{\tau A s} B B^* e^{\tau A^* s} ds$$

Reduced Equations and Backward Shooting method

Linear Case: $\lambda(t)$ is linear equation (independent of x) and u can be eliminated by $u = -\text{sign}_\epsilon(B^* \lambda)$. Thus, $x(\cdot)$ is a function of $(\lambda(1), \tau)$ by solving the state equation with $x(0) = x_0$. One can define adjoint-forward shooting method: Find $(\lambda(1), \tau)$ that satisfies

$$\Phi(\lambda(1), \tau) = (x(1) - x_1, 1 + \lambda(1) \cdot (Ax(1) + Bu(1))) = 0$$

Backward Shooting Method We use the scaled transversality condition

$$1 + \tau \lambda(1) \cdot (Ax(1) + u(\tau)Bx(1)) = 0$$

In this way one can eliminate $\tau > 0$ by this. So, we may employ the backward shooting method: find $\lambda(1)$ such that $\Phi(p(1)) = x(0) = x_0 = 0$ in which we solve the coupled system with the terminal condition $(x(1), \lambda(1))$ is given.

Theorem u_ϵ converges to u^* with minimum norm. For $0 \leq \epsilon \leq \hat{\epsilon}$

$$\tau_\epsilon \leq \tau_{\hat{\epsilon}}$$

$$\tau_\epsilon \int_0^1 |u_\epsilon|^2 dt \geq \tau_{\hat{\epsilon}} \int_0^1 |u_{\hat{\epsilon}}|^2 dt$$

$$\tau_0 \leq \tau_\epsilon \leq \tau_0 \left(1 + \frac{\epsilon}{2}\right)$$

Necessary Optimality

With coordinate change $t = \tau s$,

$$\min J_\epsilon(u, \tau) = \tau \int_0^1 \left(1 + \frac{\epsilon}{2}|u(t)|^2\right) dt$$

subject to $\frac{d}{dt}x = \tau (Ax + uBx)$, $x(0) = x_0$ and

$$u \in U_{ad} = \{u \in L^2(0, 1; R^m) : |u(t)| \leq \gamma, \text{ a.e. } \}.$$

Let $g_1(u, \tau) = Cx(1) - y = 0$ (Finite Rank), $g_2(u, \tau) = |x(1) - \bar{x}|^2 \leq \delta$

$$\min J_\epsilon(u, \tau) \quad \text{subject to } g_1(u, \tau) = 0, \quad g_2(u, \tau) \leq 0 \text{ and } u \in U_{ad}.$$

Regular Point Condition: $0 \in \text{int}\{g_u(U_{ad} - u_\epsilon) + g_\tau(R^+ - \tau_\epsilon)\}$ (3)

$$\int_0^1 (\epsilon u_\epsilon, u - u_\epsilon) dt + (g_u(u - u_\epsilon), \mu_\epsilon) \geq 0, \quad (g_\tau, \mu_\epsilon) = 0 \text{ for all } u \in U_{ad}.$$

Adjoint Equation : $-\frac{d}{dt}\lambda_\epsilon(t) = (A + uB)\lambda_\epsilon$

$$(g_\tau, \mu_\epsilon) = \int_0^1 ((A + u_\epsilon B)x_\epsilon, \lambda_\epsilon) dt.$$

$$(g_u(v), \mu_\epsilon) = \tau \int_0^1 (\lambda_\epsilon \cdot Bx_\epsilon, v) ds$$

Thus, the necessary optimality is written as

$$\int_0^1 (\epsilon u_\epsilon + \lambda_\epsilon \cdot Bx_\epsilon, u - u_\epsilon) dt \geq 0.$$

for all $u \in U_{ad}$ and

$$\int_0^1 (1 + \frac{\epsilon}{2}|u_\epsilon|^2 + (A + u_\epsilon B)x_\epsilon, p_\epsilon) dt = 0.$$

Standard Minimum Norm Problems (unconstrained):

$$\min \int_0^{\tau} |u|^2 dt$$

subject to $x(\tau) = x_1$.

$$u^*(t) = B^* e^{A^*(\tau-t)} \mu, \quad \mu = G(\tau)^{-1} (x_1 - e^{A\tau} x_0).$$

L^∞ Minimum Norm Problem minimize $\sum_{i=1}^m \gamma_i^2$ subject to

$$|u_i(t)| \leq \gamma_i \text{ on } [0, \tau], \quad \text{and} \quad x(\tau) = x_1.$$

Necessary Optimality:

$$u^* = -\text{sign}_\epsilon(\lambda(t) \cdot Bx(t)), \quad \text{and} \quad \gamma_i - \int_0^{\tau} |(\lambda(t) \cdot Bx(t))_i| dt = 0$$

Mixed Problem Min $\tau + \frac{1}{2}|\gamma|^2$

γ	δ	guess: τ	optimal τ	# of switch	control form
19.25	0.01/2	0.5	0.7236	0	1
15	0.01/2		0.7434	0	-1
11	0.01/2	1	1.0098	0	-1
10	0.01/2	1	1.0804	0	-1
8	0.01/2		1.3529	0	-1
7	0.01/2		1.1139	2	-1,1,-1
6	0.01/2		1.3406	2	-1,1,-1
5	0.01/2		1.6516	2	-1,1,-1
4	0.01/2		2.1064	2	-1,1,-1
3	0.01/2		2.8334	2	-1,1,-1

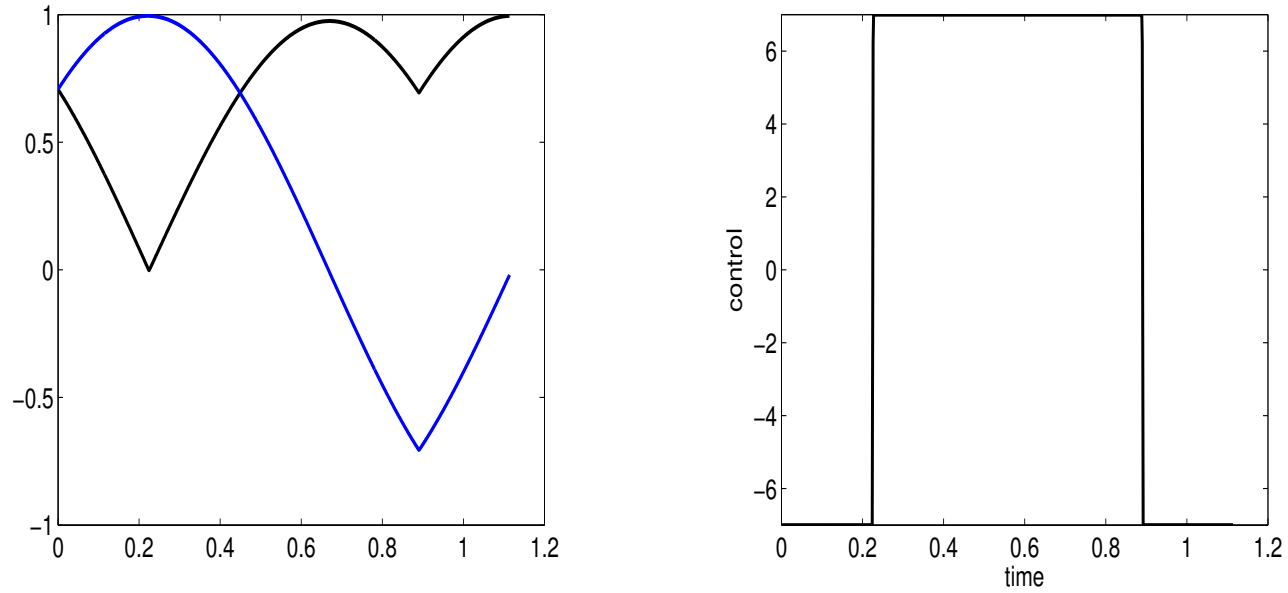


Figure 1: the Trajectory and Control with $\delta = 0.01/2$.