On Convergence of A Fixed-Point Iterate for Quasilinear Elliptic Equations

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Abstract In this paper we discuss the convergence of a fixed-point iterate for a class of quasi-linear elliptic equations that arises in variational problems for image restoration and edge detection problems. We also obtain the convergence results of a time-marching algorithm for the de-convolution in inverse scattering problems. The result can be also applied to the p-Laplacian equation by the duality method.

1 Introduction

In this paper we analyze the convergence property of the fixed point iterate (1.9) below for solving the quasilinear elliptic equation in a bounded Lipschitz domain Ω in R^d

$$-\nabla\cdot(\varphi'(|\nabla u|^2)\,\nabla u)+u=f \tag{1.1}$$

with the Neumann boundary condition $n\cdot\nabla u=0$, where $f\in L^2(\Omega)$ is given.

Our study is motivated from the problem of edge detection and image restoration [9],[3],[1] (and references therein) in the variational form, i.e.

$$\min \quad J(u)=\int_\Omega \frac{1}{2}\,(\varphi(|\nabla\, u(x)|^2)+|u(x)-f(x)|^2)\,dx \tag{1.2}$$

where the functional φ is defined by

$$\varphi(t^2)=\min_{b\geq 0}\quad\{b\,t^2+\psi(b)\} \tag{1.3}$$

where ψ is a convex function [2],[5]. Note that (1.3) is equivalent to that $s\to\varphi(s)$ is concave. In fact

$$\varphi(s)=-\max_{b\geq 0}\quad\{b\,(-s)-\psi(b)\}=-\psi^*(-s) \tag{1.4}$$

where ψ^* is the conjugate function of the convex function ψ. For example we have the following commonly used cases. The TV-type [1] is given by

$$\psi=\epsilon\,b+\frac{1}{b},\ \epsilon\geq 0\quad\to\quad\varphi(t^2)=2\,\sqrt{\epsilon+t^2}, \tag{1.5}$$

where $t \to \varphi(t^2)$ is convex. The Perona-Malik model [8] is given by

$$\psi = c\,b - log\,(b),\ c > 0 \quad \to \quad \varphi(t^2) = 1 - log\,(c + t^2), \tag{1.6}$$

where $t \to \varphi(t^2)$ is not convex. The mixed type [6] (of TV and H^1 minimization) for multi-scaled images is defined by

$$\varphi'(s) = \begin{cases} \frac{1}{\sqrt{s}} & \text{on } (0,\delta) \cup (1,\infty) \\ 1 & \text{on } (\delta, 1) \end{cases} \tag{1.7}$$

for $0 < \delta \leq .5$. Note that φ defined by (1.7) is not C^1 but is concave.

In general if for $t^2 \geq 0$, b is the unique minimizer, then we have

$$b = \varphi'(t^2) \quad \text{and} \quad t^2 = -\psi'(b). \tag{1.8}$$

Consider the fixed-point iterate

$$\begin{aligned} &-\nabla \cdot (b_k\, \nabla u_{k+1}) + u_{k+1} - \mu\,\Delta(u_{k+1} - u_k) = f \\ &b_{k+1} = \varphi'(|\nabla u_{k+1}|^2) \end{aligned} \tag{1.9}$$

Here $\mu > 0$ is arbitrary small and is used to make sure that the iterate u_k is in $H^1(\Omega)$. Otherwise we need to define the generalized solution to (1.9). If $b_k \geq \alpha > 0$ a.e. in Ω we can set $\mu = 0$. In practice we set $\mu = 0$ for the finite dimensional discretized problem.

If $t \to \varphi(t^2)$ is convex and coercive then (1.1) has a unique solution $u \in H^1(\Omega)$ and it defines the unique minimizer of (1.2). Otherwise, (1.1) may not have a solution. Thus, it is of our interests to examine the limiting properties of the sequence $u_k \in H^1(\Omega)$ generated by (1.9). We also refer to [10] for the related analysis of the iterative method (1.9).

The following is an outline of the paper. In Section 2 we discuss the convergence property of the fixed-point iterate (1.9). In order to overcome the non-existence of minimizer for the case of Perona-Malik model we introduce the regularized problems, which is motivated by [3], and analyze the corresponding fixed-point iterate. In Section 4 we consider the inverse scattering problem of the convolution type and establish the convergence of the time-marching scheme (4.3). Finally we discuss our treatment for the minimization of p-norm of $\nabla\, u$ with $p > 2$ in Section 5.

2 Convergence Analysis

In this section we analyze the convergence of the fixed-point iterate (1.9). Let $u_{k+1} \in H^1(\Omega)$ be the unique weak solution to (1.9), i.e.

$$(b_k\, \nabla\, u_{k+1}, \nabla\, \phi) + (u_{k+1}, \phi) + \mu\,(\nabla\,(u_{k+1} - u_k), \nabla\,\phi) = (f, \phi) \tag{2.1}$$

for all $\phi \in H^1(\Omega)$. Setting $\phi = u_{k+1} - u_k$ in (2.1) we obtain

$$\begin{aligned} &\tfrac{1}{2}\,(b_k, |\nabla\, u_{k+1}|^2 - |\nabla\, u_k|^2) + \tfrac{1}{2}\,((b_k + 2\mu)\,\nabla\,(u_{k+1} - u_k), \nabla\,(u_{k+1} - u_k)) \\ &\quad + \tfrac{1}{2}\,|u_{k+1}|^2 + \tfrac{1}{2}\,|u_{k+1} - u_k|^2 - \tfrac{1}{2}\,|u_k|^2 = (f, u_{k+1} - u_k). \end{aligned} \tag{2.2}$$

Since $b_k = \varphi'(|\nabla u_k|^2)$ and φ is concave, we have

$$I_k = (b_k, |\nabla\, u_{k+1}|^2 - |\nabla\, u_k|^2) - (\varphi(|\nabla\, u_{k+1}|^2) - \varphi(|\nabla\, u_k|^2), 1) \geq 0 \tag{2.3}$$

Thus, from (2.2)–(2.3)

$$J(u_{k+1}) - J(u_k) + \frac{1}{2}((b_k + 2\mu, |\nabla\,(u_{k+1} - u_k)|^2) + |u_{k+1} - u_k|^2) \leq 0 \tag{2.4}$$

Summing this in k we obtain

$$J(u_m) + \frac{1}{2}\sum_{k=1}^{m}(|u_k - u_{k-1}|^2 + (b_k + 2\mu, |\nabla\,(u_k - u_{k-1})|^2)) \leq J(u_0). \tag{2.5}$$

From (2.4)–(2.5) we conclude that

$$J(u_k) \text{ is monotonically decreasing}$$

and

$$|u_{k+1} - u_k|^2 + (b_k + 2\mu, |\nabla\,(u_{k+1} - u_k)|^2) \to 0 \tag{2.6}$$

as $k \to \infty$.

Theorem 1 Suppose φ' is bounded on R^+. Then for $\mu > 0$ we have

$$\sup_{\phi \in H^1(\Omega)} |(\varphi'(|\nabla\, u_k|^2)\,\nabla\, u_k, \nabla\,\phi) + (u_k, \phi) - (f, \phi)|/|\phi|_{H^1(\Omega)} \to 0$$

as $k \to \infty$.

Proof: The theorem follows from the fact that

$$|(b_k\,(\nabla\, u_{k+1} - \nabla\, u_k), \nabla\,\phi)| \leq \sqrt{|b_k|_\infty}|b_k^{1/2}\,(\nabla\, u_{k+1} - \nabla\, u_k)|_2 |\nabla\,\phi|_2 \to 0$$

as $k \to 0$ for all $\phi \in H^1(\Omega)$. $\square$

In general because of the lack of coercivity and convexity of $t \to \varphi(t^2)$ in (1.2), equation (1.1) may not have a solution. But we have the following theorems.

Theorem 2 Suppose $t \to \varphi(t^2)$ is convex. Then

$$\lim_{k\to\infty} J(u_k) = \inf_{u\in H^1(\Omega)} J(u).$$

Proof: First, since $J(u_k)$ is decreasing and bounded below $\lim J(u_k)$ exists. Since $t \to \varphi(t^2)$ is convex for $v \in H^1(\Omega)$, it follows from (2.1) that

(2.7)

$$J(v) - J(u_k) \geq (\varphi'(|\nabla u_k|^2)\, \nabla u_k, \nabla (v - u_k)) + (u_k - f, v - u_k)$$

$$= (b_k \nabla u_k, \nabla (v - u_k)) + (u_{k+1} - f, v - u_k) + (u_k - u_{k+1}, v - u_k)$$

$$= (b_k \nabla (u_k - u_{k+1}), \nabla (v - u_k)) - \mu (\nabla (u_{k+1} - u_k), \nabla (v - u_k)) + +(u_k - u_{k+1}, v - u_k).$$

Note that from (2.6)

$$|(b_k \nabla (u_{k+1} - u_k), \nabla (v - u_k))| \leq |b_k^{1/2} \nabla (u_{k+1} - u_k)|_2 |b_k^{1/2} \nabla (v - u_k)|_2 \to 0$$

as $k \to \infty$. From (2.6)–(2.7) we have

$$\lim_{k\to\infty} J(u_k) \leq J(v) \quad \text{for all} \;\; v \in H^1(\Omega).\square$$

Theorem 3 Suppose $|u_k|_{H^1}$ is bounded and $t \to \varphi(t^2)$ is convex. Then u_k converges weakly to the unique solution u to (1.1) in $H^1(\Omega)$.

Proof: Since u_k is a bounded sequence in the Hilbert space $H^1(\Omega)$ there exist a subsequence of u_k (denoted by the same) and $u \in H^1(\Omega)$ such that $u_k \to u$ weakly in $H^1(\Omega)$. Since $t \to \varphi(t^2)$ is convex we have the monotonicity

$$(\varphi'(|p|^2)p - \varphi'(|q|^2)q, p - q)_{R^d} \geq 0$$

and thus

(2.8) $$(\varphi'(|\nabla v|^2)\, \nabla v - \varphi'(|\nabla u_k|^2)\, \nabla u_k, \nabla v - \nabla u_k) \geq 0$$

for all $v \in H^1(\Omega)$. It follows from (2.1) and (2.8) that

$$(\varphi'(|\nabla v|^2)\, \nabla v, \nabla v - \nabla u_k) + (u_k, v - u_k) - (f, v - u_k) + \delta_k \geq 0$$

where

$$\delta_k = ((b_k + \mu)\,(\nabla u_{k+1} - \nabla u_k), \nabla v - \nabla u_k) + (u_{k+1} - u_k, v - u_k).$$

From the (2.6) and Theorem 1 we have $\delta_k \to 0$ as $k \to \infty$. Now passing limit $k \to \infty$,

$$(\varphi'(|\nabla v|^2)\, \nabla v, \nabla v - \nabla u) + (u, v - u) - (f, v - u) \geq 0.$$

For $\lambda > 0$, $\phi \in H^1(\Omega)$ we set $v = u + \lambda\,\phi$ above and upon dividing by λ we obtain

$$(\varphi'(|\nabla\,(u+\lambda\,\phi)|^2)\,\nabla\,(u+\lambda\,\phi), \nabla\,\phi) + (u+\lambda\,\phi, \phi) - (f, \phi) \geq 0.$$

By Lebesgue dominated convergence theorem and taking limit $\lambda \to 0$,

$$(\varphi'(|\nabla\,u|^2)\,\nabla\,u, \nabla\,\phi) + (u, \phi) - (f, \phi) \geq 0.$$

for all $\phi \in H^1(\Omega)$, This implies u satisfies

$$(\varphi'(|\nabla\,u|^2)\,\nabla\,u, \nabla\,\phi) + (u, \phi) - (f, \phi) = 0$$

for all $\phi \in H^1(\Omega)$. The uniqueness of the weak limit follows from the monotonicity (2.8) and thus the sequence u_k converges weakly to u in $H^1(\Omega)$. □

Corollary We further assume that

$$\int_\Omega (\varphi(|\nabla\,u(x)|^2) - \gamma\,|\nabla\,u(x)|^2)\,dx$$

is weakly lower semicontinuous for $\gamma > 0$. Then u_k converges strongly to the unique solution u to (1.1) in $H^1(\Omega)$ and thus b_k converges strongly to $\varphi'(|\nabla\,u|^2)$ in $L^2(\Omega)$.

Proof: Since $t \to \varphi(t^2)$ is convex and u_k converges weakly to u in $H^1(\Omega)$, it follows that

$$\int_\Omega \varphi(|\nabla\,u_k|^2)\,dx \to \int_\Omega \varphi(|\nabla\,u|^2)\,dx$$

as $k \to \infty$. Thus

$$\begin{aligned}
&\lim_{k\to\infty} \int_\Omega (\varphi(|\nabla\,u|^2) - \gamma\,|\nabla\,u|^2)\,dx \\
&\quad \leq \liminf_{k\to\infty} \int_\Omega (\varphi(|\nabla\,u_k|^2) - \gamma\,|\nabla\,u_k|^2)\,dx \\
&\quad \leq \int_\Omega \varphi(|\nabla\,u|^2)\,dx - \gamma\,\limsup_{k\to\infty} \int_\Omega |\nabla\,u_k|^2\,dx,
\end{aligned}$$

which implies

$$\limsup_{k\to\infty} \int_\Omega |\nabla\,u_k|^2\,dx \leq \int_\Omega |\nabla\,u|^2\,dx.$$

Combining with the weak lower semicontinuity of L^2-norm, this implies the first claim. Since φ' is bounded the last claim of the corollary follows from Lebesgue dominated convergence theorem. □

Remark For $t \to \varphi(t^2) = \sqrt{\epsilon + t^2}$ is convex and φ' is bounded. If we add an arbitrary viscosity term to φ, i.e., $\varphi_\nu(t^2) = \varphi + \nu\,t^2$, then u_k is a bounded sequence in $H^1(\Omega)$.

3 Regularization

In this section we discuss the regularized problems of (1.2) in order to remedy the difficulty concerning the nonexistence of minimizing u of problem (1.2). Following the definition (1.3) of φ, we consider the equivalent minimization

$$\min \min_{b \geq 0} \quad \frac{1}{2} \int_{\Omega} (b(x) \, |\nabla u(x)|^2 + \psi(b(x)) + |u(x) - f(x)|^2) \, dx \tag{3.1}$$

over $(u, b) \in H^1(\Omega) \times L^2(\Omega)$. We note that from (1.8) the corresponding Euler-Lagrange equation is given by

$$-\nabla \cdot (b \, \nabla u) + u = f, \quad b = \varphi'(|\nabla u|^2) \tag{3.2}$$

which is identical to (1.1). We consider the following regularized problems.

First we minimize (3.1) over $H^1(\Omega) \times K$, where K is the finite dimensional subspace of $L^2(\Omega)$ defined by

$$K = \{b \in L^2(\Omega) : b(x) = \sum_{i=1}^{m} b_i \, \chi_{\Omega_i}(x)\},$$

where Ω_i is the disjoint sets and $\Omega = \cup_{i=1}^m \Omega_i$. It is not difficult to prove that there exists a minimizing pair $(u, b) \in H^1(\Omega) \times K$ and we have

$$-\nabla \cdot (b(x) \, \nabla u(x)) + u(x) = f(x) \quad \text{wit} \;\; b_i = \varphi'(\frac{1}{mess(\Omega_i)} \int_{\Omega_i} |\nabla u|^2 \, dx). \tag{3.3}$$

Let us consider the corresponding fixed-point iterate method:

$$\begin{aligned} &-\nabla \cdot (b^k(x) \, \nabla u_{k+1}) + u_{k+1} - \mu \, \Delta(u_{k+1} - u_k) = f \\ &b_i^{k+1} = \varphi'(\frac{1}{mess(\Omega_i)} \int_{\Omega_i} |\nabla u_{k+1}|^2 \, dx). \end{aligned} \tag{3.4}$$

Next we consider the regularized version of (3.1):

$$\min \min_{b \geq 0} \quad J_\sigma(u, b) = \frac{1}{2} \int_{\Omega} ((G_\sigma * b)(x) \, |\nabla u(x)|^2 + \psi(b(x)) + |u(x) - f(x)|^2) \, dx, \tag{3.5}$$

where

$$(G_\sigma * b)(x) = \int_{\Omega} exp(-\frac{|x-y|^2}{2\sigma}) \, b(y) \, dy, \quad \sigma > 0.$$

Since G_σ is a compact linear operator on $L^2(\Omega)$ the cost functional J_σ is weakly lower semicontinuous. Thus there exists a minimizing pair $(u, b) \in H^1(\Omega) \times L^2(\Omega)$ of (3.5) and we have

$$-\nabla \cdot ((G_\sigma * b) \, \nabla u) + u = f, \quad b = \varphi'(G_\sigma * |\nabla u|^2). \tag{3.6}$$

Then the corresponding fixed-point iterate method is given by

$$(3.7)\qquad \begin{aligned} &-\nabla\cdot((G_\sigma * b^k)\,\nabla u_{k+1}) + u_{k+1} - \mu\,\Delta(u_{k+1}-u_k) = f \\ &b^{k+1} = \varphi'(G_\sigma * |\nabla u_{k+1}|^2). \end{aligned}$$

Since G_σ is compact, it follows that $(G_\sigma * b^k((x)$ is bounded below by a positive constant and thus we can set $\mu = 0$. Since

$$((G_\sigma * b^k), |\nabla\, u_{k+1}|^2 - |\nabla\, u_k|^2) = (b_k, G_\sigma * (|\nabla\, u_{k+1}|^2 - |\nabla\, u_k|^2)),$$

we have

$$((G_\sigma * b^k), |\nabla\, u_{k+1}|^2 - |\nabla\, u_k|^2) - (\varphi(G_\sigma * |\nabla\, u_{k+1}|^2) - \varphi(G_\sigma * |\nabla\, u_k|^2)) \geq 0.$$

Hence using exactly the same arguments as (2.1)–(2.3), we obtain

$$(3.8)\qquad J_\sigma(u_{k+1}, b^{k+1}) - J_\sigma(u_k, b^k) + \frac{1}{2}((G_\sigma * b_k), |\nabla\,(u_{k+1}-u_k)|^2) + |u_{k+1}-u_k|^2) \leq 0.$$

Theorem 4 Suppose φ' is bounded on R^+. The sequence (u_k, b^k) generated by (3.7) is bounded in $H^1(\Omega)\times L^2(\Omega)$ and every weak cluster point $(u,b) \in H^1(\Omega)\times L^2(\Omega)$ of (u_k, b^k) satisfies (3.6).

Proof: Since $(G_\sigma * b^k)(x)$ is bounded below by a positive constant, it follows from (3.7) that u_k is a bounded sequence in $H^1(\Omega)$ and so is b^k in $L^2(\Omega)$. Thus there exists a weak convergent subsequence $(u_{\hat{k}}, b^{\hat{k}})$ in $H^1(\Omega)\times L^2(\Omega)$. It follows from (3.8) that

$$|((G_\sigma * b^k)\,(\nabla\, u_{k+1} - \nabla\, u_k), \nabla\,\phi)| \to 0$$

as $k \to 0$ for all $\phi \in H^1(\Omega)$. Since $G_\sigma * b^{\hat{k}} \to G_\sigma * b$ strongly in $C(\Omega)$, it follows that

$$((G_\sigma * b^k)\,\nabla\, u_{k+1}, \nabla\,\phi) \to ((G_\sigma * b)\,\nabla\, u, \nabla\,\phi) \quad \text{as} \;\; k \to \infty$$

for all $\phi \in H^1(\Omega)$, which implies that the pair $(u,b) \in H^1(\Omega)\times L^2(\Omega)$ satisfies (3.6). □

Similarly, we have the following theorem concerning (3.3)–(3.4).

Theorem 5 Suppose φ' is bounded on R^+. The sequence (u_k, b^k) generated by (3.4) is bounded in $H^1(\Omega)\times L^2(\Omega)$ and every weak cluster point $(u,b) \in H^1(\Omega)\times K$ of (u_k, b^k) satisfies (3.3).

4 De-convolution

In this section we consider the de-convolution problem

$$\text{(4.1)} \qquad \min \quad J(u) = \int_\Omega \frac{1}{2}\,(|\int_\Omega K(x,y)\,u(y)\,dy - y(x)|^2 + \varphi(|\nabla u|^2))\,dx$$

where K is the symmetric positive convolution kernel. For example, in the case of eddy current testing [7] in R^2 the kernel K is given by

$$K(x,y) = \frac{1}{\sqrt{|x-y|^2 + h^2}}, \quad h > 0.$$

We define the bounded linear operator H on $L^2(\Omega)$ by

$$Hu = \int_\Omega K(x,y)\,u(y)\,dy.$$

Then, the Euler-Lagrange equation for (4.1) is given by

$$\text{(4.2)} \qquad HHu - \nabla\cdot(\varphi'(|\nabla u|^2)\,\nabla u) = Hy$$

We consider the fixed-point iterate

$$\text{(4.3)} \qquad \begin{aligned} &\frac{(u_{k+1}-u_k)}{\Delta t} - \nabla\cdot(b_k\,\nabla u_{k+1}) + HHu_k - \mu\,\Delta(u_{k+1}-u_k) = Hy \\ &b_{k+1} = \varphi'(|\nabla u_{k+1}|^2). \end{aligned}$$

It is a time-marching scheme with the explicit step for the convolution term and implicit step for the nonlinear diffusion term. For the discretized problem the most costly operation is the evaluation of the convolution Hu. We show that we can select a step size $\Delta t > 0$ such that (4.3) is convergent.

Let $u_{k+1} \in H^1(\Omega)$ be the unique weak solution to (4.3), i.e.

$$\text{(4.4)} \qquad (\frac{u_{k+1}-u_k}{\Delta t},\phi) + (b_k\,\nabla\,u_{k+1},\nabla\,\phi) + (HHu_k,\phi) + \mu\,(\nabla\,(u_{k+1}-u_k),\nabla\,\phi) = (Hy,\phi)$$

for all $\phi \in H^1(\Omega)$. Setting $\phi = u_{k+1}-u_k$ in (4.4) we obtain

$$\text{(4.5)} \qquad \begin{aligned} &\tfrac{1}{2}\,(b_k,|\nabla\,u_{k+1}|^2 - |\nabla\,u_k|^2) + \tfrac{1}{2}\,((b_k+2\mu)\,\nabla\,(u_{k+1}-u_k),\nabla\,(u_{k+1}-u_k)) \\ &\quad + \tfrac{1}{\Delta t}\,|u_{k+1}-u_k|^2 = (Hy - HHu_k, u_{k+1}-u_k). \end{aligned}$$

Here

$$\text{(4.6)} \qquad (HHu_k, u_{k+1}-u_k) = \frac{1}{2}\,|Hu_{k+1}|^2 - \frac{1}{2}\,|H(u_{k+1}-u_k)|^2 - \frac{1}{2}\,|Hu_k|^2.$$

If we choose $\Delta t > 0$ such that

$$(4.7) \qquad (\frac{1}{\Delta t}\,\phi,\phi) - \frac{1}{2}\,|H\phi|^2 + \frac{\mu}{2}\,|\nabla\phi|^2 \geq \frac{\omega}{2}\,|\phi|^2$$

for all $\phi \in H^1(\Omega)$ and some $\omega > 0$, then it follows from (2.3) and (4.5) that

$$(4.8) \qquad J(u_{k+1}) - J(u_k) + \frac{1}{2}(b_k + \mu, |\nabla\,(u_{k+1} - u_k)|^2) + \frac{\omega}{2}\,|u_{k+1} - u_k|^2 \leq 0.$$

Summing this in k we obtain

$$(4.9) \qquad J(u_m) + \frac{1}{2}\,\sum_{k=1}^{m}(\omega\,|u_k - u_{k-1}|^2 + (b_k + \mu, |\nabla\,(u_k - u_{k-1})|^2)) \leq J(u_0).$$

From (4.7)–(4.9) we conclude that

$$J(u_k) \;\text{ is monotonically decreasing}$$

and

$$\omega\,|u_{k+1} - u_k|^2 + (b_k + \mu, |\nabla\,(u_{k+1} - u_k)|^2) \to 0$$

as $k \to \infty$.

Thus, exactly the same results (Theorems 1–3 and Corollary) hold for the solutions to (4.2)–(4.3) and also the corresponding results to Theorems 4–5 hold.

5 p-Laplacian Equation

In this section for $\alpha > 2$, we consider the minimization problems

$$(5.1) \qquad \min \quad \mathcal{P}_1 = \int_\Omega (\frac{1}{\alpha}\,|\nabla\,u(x)|^\alpha - f(x)u(x))\,dx \quad \text{over } \; u \in W_0^{1,\alpha}(\Omega)$$

and

$$(5.2) \qquad \min \quad \mathcal{P}_2 = \int_\Omega (\frac{1}{\alpha}\,|\nabla\,u(x)|^\alpha + \frac{1}{2}\,|u(x) - f(x)|^2)\,dx \quad \text{over } \; u \in W^{1,\alpha}(\Omega).$$

As described in [4], the dual problem of (5.1) is given by

$$(5.3) \qquad \max \quad \mathcal{P}_1^* = \int_\Omega -\frac{1}{\beta}\,|p(x)|^\beta\,dx \quad \text{over } \; p \in L^\beta(\Omega)^n$$

subject to $\nabla \cdot p = f$, where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. It follows from Proposition 2.2 in Chapter 4 [4] that $\min \mathcal{P}_1 = \max \mathcal{P}_1^*$, and that Problem (5.1) has a unique solution u and the dual problem (5.3) has a unique solution p and we have the the extremality relation

$$(5.4) \qquad p(x) = -|\nabla\,u(x)|^{\alpha-2}\,\nabla\,u(x) \;\text{ a.e. } \; x \in \Omega.$$

In the two dimensional case ($n = 2$), there exists $\phi \in W_0^{1,\beta}(\Omega)$ such that $-\Delta\,\phi = f$. If we set $q = p + \nabla\,\phi$, then $\nabla \cdot q = 0$. Furthermore if Ω is simply connected, then there exits $\psi \in W^{1,\beta}(\Omega)$ such that

$$q = curl\,\psi = \left(\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1}\right).$$

Thus problem (5.3) is equivalent to

$$\text{(5.5)} \qquad \min \quad \int_\Omega \frac{1}{\beta}\,|\nabla\,\psi(x) + \mathrm{curl}\,\phi(x)|^\beta\,dx \quad \text{over } \; \psi \in W^{1,\beta}(\Omega).$$

The necessary and sufficient optimality of (5.5) is given by

$$\text{(5.6)} \qquad -\nabla \cdot (|\nabla\,\psi(x) + \mathrm{curl}\,\phi(x)|^{\beta-2}\,(\nabla\,\psi(x) + \mathrm{curl}\,\phi(x))) = 0$$

with the boundary condition $n \cdot \nabla\,\psi(x) = 0$ at Γ. Here we used the fact that $n \cdot \mathrm{curl}\,\phi = \tau \cdot \nabla\,\phi = 0$ at Γ. It can be shown that the fixed-point iterate for $\psi_k \in H^1(\Omega)$ satisfying $(\psi_k, 1) = 0$

$$\text{(5.7)} \qquad \begin{aligned} &(b_k\,(\nabla\,\psi_k + \mathrm{curl}\,\phi), \nabla\chi) + \mu\,(\nabla\,(\psi_{k+1} - \psi_k), \nabla\,\chi) = 0 \quad \text{for } \; \chi \in H^1(\Omega) \\ &b_{k+1}(x) = |\nabla\,\psi_{k+1}(x) + \mathrm{curl}\,\phi(x)|^{\beta-2} \;\; \text{a.e. } \; x \in \Omega \end{aligned}$$

generates the convergent sequence ψ_k to ψ, the solution to (5.6). In fact setting $\chi = \psi_{k+1} - \psi_k$ in (5.7), we obtain

$$\frac{1}{2}(b_k, |\nabla\,\psi_{k+1} + \mathrm{curl}\,\phi|^2 - |\nabla\,\psi_k + \mathrm{curl}\,\phi|^2) + (\frac{1}{2}\,b_k + \mu, |\nabla\,(\psi_{k+1} - \psi_k)|^2) = 0$$

and since $t \to t^{\frac{\beta}{2}}$ is concave,

$$(b_k, |\nabla\,\psi_{k+1} + \mathrm{curl}\,\phi|^2 - |\nabla\,\psi_k + \mathrm{curl}\,\phi|^2) \geq \frac{1}{\beta}\,(|\nabla\,\psi_{k+1} + \mathrm{curl}\,\phi|^\beta - |\nabla\,\psi_k + \mathrm{curl}\,\phi|^\beta, 1).$$

Thus Theorem 2 is applied to argue that $\psi_k \to \psi$. We then obtain from (5.4)

$$\text{(5.8)} \qquad -\nabla \cdot (c(x)\,\nabla\,u(x)) = f(x), \quad c(x) = |\nabla\,\psi(x) + \mathrm{curl}\,\phi(x)|^{\frac{\alpha-2}{\alpha-1}} \;\; \text{a.e. } \; x \in \Omega$$

which determines $u \in W_0^{1,p}(\Omega)$, the solution to problem (5.1).

The dual problem to (5.2) is given by

$$\text{(5.9)} \qquad \max \quad \mathcal{P}_2^* = \int_\Omega -\frac{1}{2}\,|\nabla \cdot p(x) - f(x)|^2 - \frac{1}{\beta}\,|p(x)|^\beta\,dx \quad \text{over } \; p \in L^\beta(\Omega)^n,$$

subject to $n \cdot p = 0$ at Γ. It can be shown [4] that $\min \mathcal{P}_2 = \max \mathcal{P}_2^*$ and (5.4) is the extremality relation. Without loss of generality we can assume that $(f, 1) = 0$. If $\phi \in$

$W^{1,q}(\Omega)$ is a solution to $-\Delta\,\phi = f,\ n\cdot\nabla\,\phi = 0$ at Γ and Ω is simply connected, then setting $q = \mathrm{curl}\,\psi - \nabla\,\phi$, we have $\nabla\cdot q = 0$ and $\psi = 0$ at Γ. Thus problem (5.9) is equivalent to

$$\min \quad \int_\Omega \frac{1}{\beta}\,|\nabla\,\psi(x) + \mathrm{curl}\,\phi(x)|^\beta\,dx \quad \text{over} \ \ \psi\in W_0^{1,\beta}(\Omega). \tag{5.10}$$

Thus, the corresponding fixed-point iterate for $\psi_k \in H_0^1(\Omega)$ is given by

$$\begin{aligned} &(b_k\,(\nabla\,\psi_k + \mathrm{curl}\,\phi), \nabla\chi) + \mu\,(\nabla\,(\psi_{k+1}-\psi_k), \nabla\,\chi) = 0 \quad \text{for} \ \ \chi\in H_0^1(\Omega) \\ &b_{k+1}(x) = |\nabla\,\psi_{k+1}(x) + \mathrm{curl}\,\phi(x)|^{\beta-2} \ \ \text{a.e.} \ \ x\in\Omega, \end{aligned} \tag{5.11}$$

and it can be shown that ψ_k converges to ψ, the solution to (5.10). Then the solution $u\in W^{1,p}(\Omega)$ to problem (5.2) is determined by

$$\begin{aligned} &-\nabla\cdot(c(x)\,\nabla\,u(x)) + u(x) = f(x) \ \ \text{a.e.} \ \ x\in\Omega \quad \text{with} \ \ n\cdot\nabla u = 0 \ \ \text{at} \ \ \Gamma \\ &c(x) = |\nabla\,\psi(x) + \mathrm{curl}\,\phi(x)|^{\frac{\alpha-2}{\alpha-1}} \ \ \text{a.e} \ \ x\in\Omega. \end{aligned} \tag{5.12}$$

References

[1] P.Blomgren, T.F.Chan and P.Mullet, Extensions to total variation denoising, F.T.Luk ed., Advanced signal processing, algorithms, architectures, and implementation VII, SPIE Vol.3162, (1997).

[2] P.Charbonnier, L.Blanc-Feraud, G.Aubert and M.Barlaud, Deterministic edge-preserving regularization in computed imaging, Research Note 94-01, Laboratoire Informatique Signaux et Systemes de Sophia Antipolis, Universite de Nice-Sophia Antipolis. 1995.

[3] F.Catte, P.L.Lions, J.M.Morel and T.Coll, Image selective smoothing and edge detection by nonlinear diffusion, SIAM J.Numer.Anal., 29 (1992), 182-193.

[4] I.Ekeland and R.Temam, *Convex Analysis and Variational Problems, North-Holland, New York, (1976).*

[5] S.Geman and G.Reynolds, Constrained restoration and the recovery of discontinuities, IEEE Trans. Pattern Analysis and Machine Intelligence, 14 (1992), 367-383.

[6] K.Ito and K.Kunisch, Image restoration with mixed TV and H^1 regularization, preprint.

[7] *K.Ito and F.Kojima, Inverse electromagnetic analysis using Tikhonov regularization with sub-differential norm, preprint.*

[8] *P.Perona and J.Malik, Scale-space and edge detection using anisotropic diffusion, IEEE PAMI, 12 (1990), 629-639.*

[9] *J.Weickert, Anisotropic Diffusion in Image Processing, European Consortium for Mathematics in Industry, B.G.Teubner Stuttgart, (1998).*

[10] *E.Zeidler Nonlinear Functional Analysis and Its Applications II/B, Nonlinear Monotone Operators, Springer-Verlag, New York (1985).*