Error Estimates for Viscosity Solutions of Hamilton–Jacobi Equation under Quadratic Growth Conditions

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Abstract In this paper we develop a comparison lemma for viscosity solutions for the Hamilton–Jacobi equations. We consider locally Lipschitz solutions with quadratic growth and assume the quadratic growth of the Hamiltonian. An error estimate of viscosity solutions using the weighted sup norm is obtained.

1 Introduction

In this paper we discuss the uniqueness and error estimate for viscosity solutions of Hamilton–Jacobi equations in the class of locally Lipschitz continuous functions with quadratic growth. Such problems are motivated from Hamilton–Jacobi equations that arise in optimal control problems (e.g. [FS],[Mc]) with a quadratic cost. That is, we consider the equation in R:

$$-\left[-x\,V_x+\frac{1}{2\gamma^2}\,|V_x|^2+\frac{\sigma^2}{2}\,|x|^2\right]=0,\quad x\in R,\quad V(0)=0.$$

Then it has the two smooth solutions

$$V^{\pm}(x)=\frac{1}{2}\,(\gamma^2\pm\sqrt{\gamma^2\,(\gamma^2-\sigma^2)})\,|x|^2$$

and infinitely many viscosity solutions provided that $\gamma>\sigma$. The objective of this paper is to develop the comparison principle for viscosity solutions which are locally Lipschitz with quadratic growth. For example, we will prove that V^- is a unique solution to the above in the class Σ;

$$\Sigma=\{u\in C(R):|u(x)-u(y)|\le c_2\,r|x-y|\ \text{ for }\ |x|,\ |y|\le r\}$$

where $0\le c_2<2$. The question of uniqueness for viscosity solutions of Hamilton–Jacobi equations has been considered in a number of papers, particularly by Ishii [Is], Crandall-Ishii-Lions [CIL] and McEneaney [Mc].

The result in this paper is new and improves the existing results in the following manner. We establish the error estimate of the form

$$\sup_{x\in R^n}\ \frac{1}{c+|x|^2}\,(u(x)-v(x))^+,\ \ c>0$$

in terms of the problem data. Here, u, v are solutions to the Hamilton-Jacobi equation for the Cauchy problem ($u = u(t, x)$)

$$u_t + H(t, x, u, u_x) = 0, \quad u(0, x) = u_0(x) \tag{1.1}$$

in $[0, \tau) \times R^n$ and as well as for the stationary problem

$$u + H(x, u, u_x) = 0. \tag{1.2}$$

Our approach is motivated by the one of [Mc]. We consider the class Σ_{c_1,c_2} of solution u that are continuous and locally Lipschitz in x satisfying

$$|u(t, x) - u(t, y)| \leq (c_1 + c_2\, r)\, |x - y| \tag{1.3}$$

for $t \in [0, \tau] \times B_r$ where $B_r = \{x \in R^n : |x| \leq r\}$. We assume that there exists constants a_0, $a_1 \leq 0$ and $a_2 \geq 0$ and a function $a \in C(R^n)^n$ such that

$$H(t, x, v, q) - H(t, x, u, p) \leq a_0\, (u - v) + (a(x), p - q) + (a_3 + a_2(c_1 + c_2\, r))\, |p - q| \tag{1.4}$$

for all $t \in [0, \tau]$, $x \in B_r$, $u \geq v$, and $|p|$, $|q| \leq (c_1 + c_2\, r)$, where $(a(x), x) \leq a_1 |x|^2$ for $x \in R^n$. We use the comparison function of the form

$$\Phi(x, y) = \frac{1}{c + |x|^2}\, u(x) - \frac{1}{c + |y|^2}\, v(y) - \frac{|x - y|^2}{\epsilon}.$$

which is different from the one used in [Mc]. Using this comparison function we are able to establish the error estimate; assume that u, $v \in \Sigma_{c_1,c_2}$, u is a subsolution of $u_t + H(t, x, u, u_x) = f$ and v is a supersolution of $v_t + H(t, x, v, v_x) = g$. Then, for any $\varepsilon > 0$ there exists a constant $\bar{c} > 0$ (depends on ε and $c_1 a_2 + a_3$) such that for $c \geq \bar{c}$

$$\sup_{R^n} \frac{(u(t, x) - v(t, x))^+}{c + |x|^2} \leq e^{\omega t} \sup_{R^n} \frac{(u_0 - v_0)^+}{c + |x|^2} + \int_0^t e^{\omega(t-s)} \sup_{R^n} \frac{(f(s, x) - g(s, x))^+}{c + |x|^2}\, ds \tag{1.5}$$

where $\omega = a_0 + 2\, \max(0, a_1 + c_2 a_2) + \varepsilon$.

In [Mc] the following comparison result is proved. Suppose $V(t, x)$, $W(t, x)$ are continuous and uniformly locally Lipschitz in x with

$$|V(t, x) - V(t, y)| \leq K_R(V)\, |x - y|, \quad |W(t, x) - W(t, y)| \leq K_R(W)\, |x - y|,$$

for all x, $y \in Q_T^R = [0, T] \times \{x \in R^n : |x| \leq R\}$. Let W be a viscosity subsolution to (1.1) and V be a viscosity supersolution to (1.1). Assume there exists a constant $\alpha < \infty$ such that for all $R < \infty$ there exists a k_R such that if there exists a $(t, x) \in Q_T^R$ such that $W(t, x) > V(t, x)$, then

$$H(t, x, V(t, x), p) - H(t, x, W(t, x), q) \leq k_R\, (W(t, x) - V(t, x)) + \alpha(1 + R)\, |p - q| \tag{1.6}$$

for all $p,\ q \in R^n$ such that $|p|,\ |q| \le \max\{K_R(V), K_R(W)\}$. Then $W \le V$ if $W \le V$ at $t = 0$.

If $V,\ W \in \Sigma_{c_1,c_2}$, then our assumption is similar to (1.6) except the term $(a(x), p-q)$. This term allows us to have a shaper estimate of the growth constant ω in (1.5) and the inclusion of a drift term $a(x)\cdot p$ with $a \in C^1(R^n)$ in the Hamiltonian H. The estimate (1.5) gives not only the comparison result but also the error estimate and continuity result of viscosity solutions in class Σ_{c_1,c_2}. We also discuss the stationary problem (1.2) and the term $(a(x), p-q)$ plays the more essential role in establishing the error estimate. An extension to classes of polynomial growth at ∞ is of important but is not discussed in this paper. We restricted our analysis for class Σ_{c_1,c_2} since the nonlinear regulator problem discussed below as well as the corresponding differential game can be included in the class.

The existence of viscosity solutions in class Σ_{c_1,c_2} to Hamilton-Jacobi equation (1.1) and (1.2) under quadratic growth conditions (2.1) and (3.1) on the Hamiltonian H has been discussed for example [BFN],[MI],[It]. The Hamilton-Jacobi equation we consider here is, for example, motivated from the optimal control problem;

$$\text{(1.7)} \qquad \min \quad J(s,y;u) = \int_s^T f^0(t,x(t),u(t))\,dt + g(x(T))$$

subject to the control system

$$\text{(1.8)} \qquad \frac{d}{dt}x(t) = f(t,x(t),u(t)),\ \ t > s \quad \text{with} \quad x(s) = y; \ \text{ and } \ u(t) \in U$$

where U is a closed convex set in R^m. We consider the case when the performance index $f^0(t,x,u)$ has the quadratic growth both in (x,u) and g is of quadratic growth at infinity. Under appropriate conditions on C^1 functions $f : R \times R^n \times R^m \to R^n$ and $f^0 : R \times R^n \times R^m \to R$ and $g : R^n \to R^n$ there exists an optimal control to problem (1.7)–(1.8) and the value function $V : R \times R^n \to R$ defined by

$$V(T-s,y) = \inf\ J(s,y;u) \ \text{ subject to } \ (1.8)$$

over $u \in L^1(s,T;R^m)$, is a viscosity solution to (1.1) with $V(0,x) = g(x)$. The Hamiltonian H is given by

$$H(t,x,u,p) = \sup_{u\in U}\,\{-(p, f(t,x,u)) - f^0(t,x,u)\}.$$

Let $f^0(t,x,u) = \ell(t,x) + \frac{1}{2}\,|u|^2$, $U = R^m$. Thus, (1.5) implies that

$$\sup_{x\in R^n}\ \frac{1}{c+|x|^2}\,|V^1(t,x) - V^2(t,x)| \le e^{\omega t}\,(\sup_{x\in R^n}\ \frac{1}{c+|x|^2}\,|g^1(x) - g^2(x)|$$

$$+\int_0^t (\sup_{x\in R^n}\ \frac{1}{c+|x|^2}\,|\ell^1(s,x) - \ell^2(s,x)|\,ds)$$

for appropriately chosen $c > 0$ and $\omega \in R^+$, where $V^i \in \Sigma_{c_1,c_2}$ are the viscosity solutions to (1.1) (corresponding to the data (g^i, ℓ^i) appearing in problem (1.7)).

We conclude this section by recalling the definition of viscosity solution. We consider the first order PDE of the form

$$F(y, u, u_y) = 0 \quad \text{in} \ \ \Omega. \tag{1.9}$$

We state the definition of the viscosity solution [CL],[CEL] of (1.9).

Definition 1.1: A function $\varphi(y) \in C(\Omega)$ is a *subsolution* of (1.9) provided that for all $\psi \in C^1(\Omega)$, if $\varphi - \psi$ attains a (local) maximum at $y \in \Omega$, then

$$F(y, \varphi(y), D\psi(y)) \leq 0.$$

A function $\varphi(y) \in C(\Omega)$ is is a *suppersolution* of (1.9), if $\varphi - \psi$ attains a (local) minimum at $y \in \Omega$, then

$$F(y, \varphi(y), D\psi(y)) \geq 0.$$

A function $\varphi \in C(\Omega)$ is a *viscosity solution* of (1.1) if it is supper and sub solution of (1.9).

2 Stationary Problem

In this section we consider the stationary equation (1.2)

$$u + H(x, u, u_x) = 0 \quad \text{in} \ \ R^n.$$

We assume that H is continuous and that there exists constants $a_0,\ a_1 \leq 0$ and $a_2,\ a_3 \geq 0$ and a function $a \in C(R^n)^n$ such that

$$H(x, v, q) - H(x, u, p) \leq a_0\,(u - v) + (a(x), p - q) + (a_3 + a_2(c_1 + c_2\,r))\,|p - q| \tag{2.1}$$

for all $x \in B_r$, $u \geq v$, and $|p|,\ |q| \leq (c_1 + c_2\,r)$, where $(a(x), x) \leq a_1\,|x|^2$ for $x \in R^n$.

First, we note that if $u \in \Sigma_{c_1,c_2}$ then for each $c > 0$ $\dfrac{u(x)}{c + |x|^2}$ is Lipschitz continuous in x. In fact, we have

$$\begin{aligned} &\left|\frac{u(x)}{c + |x|^2} - \frac{u(y)}{c + |y|^2}\right| \\ &\leq \frac{|u(x) - u(y)|}{c + |x|^2} + \frac{|u(y)|}{c + |y|^2}\,\frac{|x| + |y|}{c + |x|^2}\,|x - y| \leq M\,|x - y| \end{aligned} \tag{2.2}$$

for some $M > 0$ such that $\dfrac{c_1 + c_2 s}{c + s^2}(1 + \dfrac{2s}{c + s^2}) \leq M$ for $s \in R^+$.

Theorem 2.1 *Assume that* $u\ , v \in \Sigma_{c_1,c_2}$*,* u *is a subsolution of* $u + H(x, u, u_x) = f(x)$ *and* v *is a supersolution of* $v + H(x, v, v_x) = g(x)$*. For* $\delta > 0$ *let*

$$\psi(x) = \frac{1}{c + |x|^{2+\delta}}.$$

Then, for any $\varepsilon > 0$ there exists a constant $\bar{c} > 0$ (depends on ε and $c_1 a_2 + a3$) such that for $c \geq \bar{c}$

$$(1 - a_0 - (2+\delta)\,\max(0, a_1 + c_2 a_2) - \varepsilon)\sup_{R^n}\ \psi(u-v)^+ \leq \sup_{R^n}\ \psi(f-g)^+.$$

Proof: It follows from (2.2) that if $u,\ v \in \Sigma_{c_1,c_2}$ then

$$\lim_{|x|\to\infty}\ \psi(x)u(x) = \lim_{|x|\to\infty}\ \psi(x)v(x) = 0. \tag{2.3}$$

We choose a function $\beta \in C^\infty(R^n)$ satisfying

$$0 \leq \beta \leq 1,\ \ \beta(0) = 1,\ \ \beta(x) = 0 \text{ if } |x| > 1. \tag{2.4}$$

Let

$$M = \max(\sup_{R^n}\ \psi(x)|u(x)|, \sup_{R^n}\ \psi(x)|v(x)|).$$

Define the function $\Phi : R^n \times R^n \to R$ by

$$\Phi(x,y) = \psi(x)u(x) - \psi(y)v(y) + 3M\beta_\epsilon(x-y) \tag{2.5}$$

where

$$\beta_\epsilon(x) = \beta(\frac{x}{\epsilon}) \quad \text{for}\ \ x \in R^n. \tag{2.6}$$

Off the support of $\beta_\epsilon(x-y)$, $\Phi \leq 2M$, while if $|x| + |y| \to \infty$ on this support, then $|x|,\ |y| \to \infty$ and thus from (2.3) $\lim_{|x|+|y|\to\infty}\ \Phi \leq 3M$. We may assume that $u(\bar{x}) - v(\bar{x}) > 0$ for some $\bar{x}$. Then,

$$\Phi(\bar{x},\bar{x}) = \psi(\bar{x})(u(\bar{x}) - v(\bar{x})) + 3M\ \beta_\epsilon(0) > 3M.$$

Hence Φ attains its maximum value at some point $(x_0, y_0) \in R^n \times R^n$. Moreover, $|x_0 - y_0| \leq \epsilon$ since $\beta_\epsilon(x_0 - y_0) > 0$. Now we let x_0 be a maximum point of

$$\psi(x)\left(u(x) - \frac{\psi(y_0)v(y_0) - 3M\beta_\epsilon(x-y_0) + \Phi(x_0,y_0)}{\psi(x)}\right).$$

Since $\psi > 0$ and

$$\psi(y_0)v(y_0) - 3M\beta_\epsilon(x_0 - y_0) + \Phi(x_0,y_0) = \psi(x_0)u(x_0),$$

the function

$$x \to u(x) - \frac{\psi(y_0)v(y_0) - 3M\beta_\epsilon(x-y_0) + \Phi(x_0,y_0)}{\psi(x)}$$

attains a maximum 0 at x_0. Since u is a subsolution, thus

$$\psi(x_0)(u(x_0) + H(x_0, u(x_0), p)) \leq \psi(x_0)f(x_0).$$

(2.7)

$$\text{with}\ \ p = (2+\delta)\psi(x_0)u(x_0)|x_0|^\delta\, x_0 - \frac{3M\beta'_\epsilon(x_0 - y_0)}{\psi(x_0)},$$

where we used the fact that $(|x|^{2+\delta})' = (2+\delta)|x|^{\delta}x$. Moreover, since $u \in \Sigma_{c_1,c_2}$

$$|p| \leq c_1 + c_2\,|x_0|. \tag{2.8}$$

Similarly, the function

$$y \to v(y) - \frac{\psi(x_0)u(x_0) + 3M\beta_\epsilon(x_0 - y) - \Phi(x_0, y_0)}{\psi(y)}$$

attains a minimum 0 at y_0 and since v is a super solution

$$\begin{gathered}\psi(y_0)(v(y_0) + H(y_0, v(y_0), q)) \geq \psi(y_0)g(y_0). \\ \text{with } \; q = (2+\delta)\psi(y_0)v(y_0)|y_0|^{\delta}\, y_0 - \frac{3M\beta_\epsilon'(x_0 - y_0)}{\psi(y_0)}\end{gathered} \tag{2.9}$$

where $|q| \leq c_1 + c_2\,|y_0|$. Thus by (2.7) and (2.9) we have

$$\begin{aligned}&\psi(x_0)u(x_0) - \psi(y_0)v(y_0) \\ &\quad \leq \psi(y_0)H(y_0, v(y_0), q) - \psi(x_0)H(x_0, u(x_0), p) + \psi(x_0)f(x_0) - \psi(y_0)g(y_0)\end{aligned} \tag{2.10}$$

Since $\Phi(x_0, y_0) \geq \Phi(\bar{x}, \bar{x})$, we have

$$\psi(x_0)u(x_0) - \psi(y_0)v(y_0) \geq \psi(\bar{x})(u(\bar{x}) - v(\bar{x})) + 3M\,(1 - \beta_\epsilon(x_0 - y_0))$$

and thus

$$(\psi(x_0) - \psi(y_0))\,u(x_0) + \psi(y_0)(u(x_0) - v(y_0)) \geq \psi(\bar{x})(u(\bar{x}) - v(\bar{x})) + 3M\,(1 - \beta_\epsilon(x_0 - y_0)).$$

Since

$$\begin{aligned}|(\psi(x_0) - \psi(y_0))\,u(x_0)| \;&\leq\; \psi(x_0)|u(x_0)|\frac{(2+\delta)(|x_0|^{1+\delta} + |y_0|^{1+\delta})}{c + |y_0|^{2+\delta}}\,|x_0 - y_0| \\ &\leq\; const\,|x_0 - y_0|,\end{aligned} \tag{2.11}$$

it follows that $u(x_0) \geq v(y_0)$ for sufficiently small $\epsilon > 0$. Note that

$$\begin{aligned}\psi(y_0)H(y_0, v(y_0), q) - \psi(x_0)H(x_0, u(x_0), p) &= (\psi(y_0) - \psi(x_0))H(x_0, u(x_0), p) \\ &+\psi(y_0)(H(y_0, u(x_0), p) - H(x_0, u(x_0), p)) + \psi(y_0)(H(y_0, v(y_0), q) - H(y_0, u(x_0), p)).\end{aligned}$$

From (2.1) and (2.10) we have that

$$\begin{aligned}&\psi(x_0)u(x_0) - \psi(y_0)v(y_0) - (\psi(x_0)f(x_0) - \psi(y_0)g(y_0)) \\ &\quad \leq O(\epsilon) + \psi(y_0)(a_0(u(x_0) - v(y_0)) + (a(y_0), p - q) + (a_3 + a_2(c_1 + c_2 r))|p - q|)\end{aligned}$$

and from (2.11)

$$\psi(x_0)u(x_0) - \psi(y_0)v(y_0) - (\psi(x_0)f(x_0) - \psi(y_0)g(y_0))$$

$$\le O(\epsilon) + a_0(\psi(x_0)u(x_0) - \psi(y_0)v(y_0)) \tag{2.12}$$

$$+\psi(y_0)((a(y_0), p-q) + (a_3 + a_2(c_1 + c_2 r))|p-q|)$$

where $r = \max\{|x_0|, |y_0|\}$ and $O(\epsilon) \to 0$ as $\epsilon \to 0$. Now we evaluate $p - q$, i.e.,

$$p - q = (2+\delta)(\psi(x_0)u(x_0) - \psi(y_0)v(y_0))|y_0|^\delta\, y_0$$

$$+(2+\delta)\psi(x_0)u(x_0)(|x_0|^\delta x_0 - |y_0|^\delta y_0) + 3M\beta_\epsilon'(x_0 - y_0)\,(|x_0|^{2+\delta} - |y_0|^{2+\delta}).$$

Since

$$|\psi(x_0)u(x_0)|x_0|^\delta\, x_0| \le \left|\frac{u(x_0)}{c + |x_0|^2}\right| \frac{|x_0|^\delta(c + |x_0|^2)}{c + |x_0|^{2+\delta}}\,|x_0| \le M_1\,|x_0|$$

for some $M_1 > 0$, it follows from (2.8) that

$$|\beta_\epsilon'(x_0 - y_0)(c + |x_0|^{2+\delta})| \le M_2\,(1 + |x_0|)$$

for some $M_2 > 0$. Thus,

$$|3M\beta_\epsilon'(x_0 - y_0)\,(|x_0|^{2+\delta} - |y_0|^{2+\delta})| \le 3(2+\delta)MM_2\frac{r^{1+\delta}(1+r)}{c + r^{2+\delta}}|x_0 - y_0|,$$

and therefore

$$(2+\delta)\psi(x_0)u(x_0)(|x_0|^\delta x_0 - |y_0|^\delta y_0) + 3M\beta_\epsilon'(x_0 - y_0)\,(|x_0|^{2+\delta} - |y_0|^{2+\delta}) = O(\epsilon).$$

Thus, in the right-hand side of (2.12) we have

$$\psi(y_0)((a(y_0), p-q) + (a_3 + a_2(c_1 + c_2 r))\,|p-q|)$$

$$\le O(\epsilon) + (2+\delta)\frac{a_1\,|y_0|^{2+\delta} + (a_3 + a_2(c_1 + c_2|y_0|))|y_0|^{1+\delta}}{c + |y_0|^{2+\delta}}\,(\psi(x_0)u(x_0) - \psi(y_0)v(y_0)).$$

Hence from (2.12) we conclude

$$\omega_{y_0}\,(\psi(x_0)u(x_0) - \psi(y_0)v(y_0)) \le \psi(x_0)f(x_0) - \psi(y_0)g(y_0) + O(\epsilon) \tag{2.13}$$

where

$$\omega_{y_0} = 1 - a_0 - (2+\delta)\,\frac{a_1\,|y_0|^{2+\delta} + (a_3 + a_2(c_1 + c_2|y_0|))|y_0|^{1+\delta}}{c + |y_0|^{2+\delta}}.$$

For any $\varepsilon > 0$ there exists a constant $\bar{c} = \bar{c}(\varepsilon, c_1 a_2 + a_3) > 0$ such that for $c \ge \bar{c}$

$$\omega_{y_0} \le \omega = 1 - a_0 - (2+\delta)\,\max(0, a_1 + c_2 a_2) - \varepsilon$$

and thus

$$\omega\left(\psi(x_0)u(x_0) - \psi(y_0)v(y_0)\right) \le \psi(x_0)f(x_0) - \psi(y_0)g(y_0) + O(\epsilon). \tag{2.14}$$

Assume that $\omega > 0$. For $x \in R^n$ we have

$$\psi(x)(u(x) - v(x)) + 3M = \Phi(x,x) \le \Phi(x_0,y_0) \le \psi(x_0)u(x_0) - \psi(y_0)v(y_0) + 3M$$

and so by (2.14)

$$\begin{aligned}\omega \sup_{R^n} \psi(x)(u(x)-v(x))^+ &\le \omega\left(\psi(x_0)u(x_0) - \psi(y_0)v(y_0)\right) \le \psi(x_0)f(x_0) - \psi(y_0)g(y_0) + O(\epsilon)\\ &\le \sup_{R^n} \psi(f-g)^+ + |\psi(x_0)g(x_0) - \psi(y_0)g(y_0)| + O(\epsilon)\\ &\le \sup_{R^n} \psi(f-g)^+ + \omega_{\psi g}(\epsilon) + O(\epsilon)\end{aligned}$$

where $\omega_{\psi g}(\cdot)$ is the modulus of continuity of ψg. Now the claim follows by letting $\epsilon \to 0$. □

Letting $\delta \to 0^+$ we obtain the following theorem.

Theorem 2.2 *Assume that $u,\ v \in \Sigma_{c_1,c_2}$, u is a subsolution of $u + H(x,u,u_x) = f(x)$ and v is a supersolution of $v + H(x,v,v_x) = g(x)$. Then, for any $\varepsilon > 0$ there exists a constant $\bar{c} > 0$ (depends on ε and $c_1a_2 + a_3$) such that for $c \ge \bar{c}$*

$$(1 - a_0 - 2\max(0, a_1 + c_2a_2) - \varepsilon) \sup_{R^n} \frac{(u-v)^+}{c+|x|^2} \le \sup_{R^n} \frac{(f-g)^+}{c+|x|^2}.$$

Proof: It suffices to prove that

$$\eta_\delta = \sup_{R^n} \frac{u}{c+|x|^{2+\delta}} \to \sup_{R^n} \frac{u}{c+|x|^2} = \eta$$

as $\delta \to 0^+$ for $u \in C(R^n)$. Since $\eta_\delta \le \eta$, it follows that $\lim \eta_{\delta_n} = \bar{\eta} \le \eta$ for all convergent subsequence η_{δ_n}. Suppose $\eta - \bar{\eta} = \epsilon > 0$. Then there exists a $\bar{x} \in R^n$ such that

$$\frac{u(\bar{x})}{c+|\bar{x}|^2} \ge \eta - \frac{\epsilon}{2}.$$

Since

$$\eta_{\delta_n} \ge \frac{u(\bar{x})}{c+|\bar{x}|^{2+\delta}} \to \frac{u(\bar{x})}{c+|\bar{x}|^2} \quad \text{as } \delta_n \to 0^+,$$

we obtain $\bar{\eta} \ge \eta - \frac{\epsilon}{2}$, which contradicts the assumption. Thus, $\bar{\eta} = \eta$ and $\lim\ \eta_\delta = \eta$. □

If $u\ , v \in \Sigma_{c_1,c_2}$ are viscosity solutions to $u + H(x,u,u_x) = f(x)$ and $v + H(x,v,v_x) = g(x)$, respectively, then it follows from Theorem 2.2 that

$$\omega \sup_{R^n} \frac{|u-v|}{c+|x|^2} \le \sup_{R^n} \frac{|f-g|}{c+|x|^2}.$$

In particular this implies the uniqueness of viscosity solutions to (1.2) in the class Σ_{c_1,c_2}.

In order to apply Theorem 2.2 we must have $1-a_0>0$. But we have $1-a_0=0$ for the example in Introduction and thus we cannot apply the theorem directly to prove that $V^-(x)$ is the unique solution in the class Σ. The following corollary utilizes the fact that if $a_1<0$ then $a_1+a_2c_2$ can be negative to extend the theorem to solutions in the class Σ_{0,c_2}. As a consequence we can show our claim for the example in Introduction.

Corollary 2.3 *Assume that $u,\ v\in\Sigma_{0,c_2}$ (i.e., $c_1=0$), u is a subsolution of $u+H(x,u,u_x)=0$ and v is a supersolution of $v+H(x,v,v_x)=0$. Suppose that $c_1=0$ and $a_3=0$ in (2.1) and the functions*

$$\frac{u(x)}{|x|^2},\quad \frac{v(x)}{|x|^2},\quad \frac{f(x)}{|x|^2},\quad \textit{and}\quad \frac{g(x)}{|x|^2} \tag{2.15}$$

are continuous at $x=0$ (consequently, $u(0)=v(0)=f(0)=g(0)=0$). Then, we have

$$(1-a_0-2\,(a_1+c_2a_2))\sup_{R^n}\frac{(u-v)^+}{|x|^2}\le\sup_{R^n}\frac{(f-g)^+}{|x|^2}.$$

Proof: In the proof of Theorem 2.1 we set ψ such that

$$\psi_\epsilon=\frac{1}{\sqrt{\epsilon}+q(|x|^2)}$$

where an increasing function $q\in C^1(R)$ satisfies

$$q(s)=s\ \text{ if }\ s\le 1\quad\text{and}\quad q(s)=|s|^{1+\frac{\delta}{2}}\ \text{ if }\ s\ge 2.$$

Then, by the assumption

$$M=\max\{\sup_{R^n}\ \psi_\epsilon(x)|u(x)|,\sup_{R^n}\ \psi_\epsilon(x)|v(x)|\}$$

is bounded uniformly in $\epsilon>0$ and Φ is continuous. Let $(x_\epsilon,y_\epsilon)\in R^n\times R^n$ be a maximizing point of Φ. Using exactly the same arguments as in the proof of Theorem 2.1 we have

$$\omega_{y_\epsilon}\left(\psi_\epsilon(x_\epsilon)u(x_\epsilon)-\psi_\epsilon(y_\epsilon)v(y_\epsilon)\right)\le\psi_\epsilon(x_\epsilon)f(x_\epsilon)-\psi_\epsilon(y_\epsilon)g(y_\epsilon)+O(\epsilon)$$

where

$$\omega_{y_\epsilon}=1-a_0-(a_1+a_2c_2)\frac{2q'(|y_\epsilon|^2)|y_\epsilon|^2}{\sqrt{\epsilon}+q(|y_\epsilon|^2)}.$$

Letting $\epsilon\to 0^+$, we obtain

$$\omega_\delta\ \sup_{R^n}\ \frac{(u-v)^+}{q(|x|^2)}\le\sup_{R^n}\ \frac{(f-g)^+}{q(|x|^2)}$$

where

$$\omega_\delta=1-a_0-(a_1+a_2c_2)\sup_s\frac{2q'(s)s}{q(s)}$$

and then letting $\delta \to 0^+$ we obtain the desired estimate. □

Now, let us apply the corollary to the example in Introduction. For the example $c_1 = 0$, $a_3 = 0$, and $1 - a_0 = 0$, $a_1 = -1$ and $a_2 = \frac{1}{\gamma^2}$. Thus if $c_2 < \gamma^2$, then $\omega = 1 - a_0 - 2(1 - a_2 c_2) > 0$. Hence it follows from Corollary 2.3 that $V^-(x)$ is the unique solution in the class $\Sigma = \Sigma_{0,c_2}$ satisfying (2.15).

3 Cauchy Problem

Next we consider the Cauchy problem (1.1)

$$u_t + H(t, x, u, u_x) = 0, \quad u(0, x) = u_0(x)$$

in $\Omega = [0, \tau) \times R^n$. We assume that H is continuous and that there exists constants a_0, $a_1 \le 0$ and $a_2 \ge 0$ and a function $a \in C(R^n)^n$ such that

$$H(t, x, v, q) - H(t, x, u, p) \le a_0\,(u - v) + (a(x), p - q) + (a_3 + a_2(c_1 + c_2\,r))\,|p - q| \tag{3.1}$$

for all $t \in [0, \tau]$, $x \in B_r$, $u \ge v$, and $|p|$, $|q| \le (c_1 + c_2\,r)$, where $(a(x), x) \le a_1|x|^2$.

Theorem 3.1 *Assume that u, $v \in \Sigma_{c_1,c_2}$, u is a subsolution of $u_t + H(t, x, u, u_x) = f$ and v is a supersolution of $v_t + H(t, x, v, v_x) = g$. For $\delta > 0$ let*

$$\psi(x) = \frac{1}{c + |x|^{2+\delta}}$$

Then, for any $\varepsilon > 0$ there exists a constant $\bar{c} > 0$ (depends on ε and $c_1 a_2 + a_3$) such that for $c \ge \bar{c}$

$$\sup_{R^n}\ \psi(x)(u(t, x) - v(t, x))^+ \le e^{\omega t} \sup_{R^n}\ \psi(u_0 - v_0)^+ + \int_0^t e^{\omega(t-s)} \sup_{R^n}\ \psi(f(s, x) - g(s, x))^+\, ds$$

where $\omega = a_0 + (2 + \delta)\max(0, a_1 + c_2 a_2) + \varepsilon$.

Proof: For ω and $c > 0$ define

$$F(t) = \int_0^t e^{-\omega\sigma} \sup_{R^n}\ \psi(x)(f(\sigma, x) - g(\sigma, x))^+\, d\sigma.$$

We assume that

$$e^{-\omega\bar{t}} \sup_{R^n}\ \psi(x)(u(\bar{t}, x) - v(\bar{t}, x))^+ - \sup_{R^n}\ \psi(x)(u_0(x) - v_0(x))^+ - F(\bar{t}) = \alpha > 0 \tag{3.2}$$

Then there exists an $\bar{x} \in R^n$ such that

$$e^{-\omega\bar{t}} \psi(\bar{x}) u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{x})) - \sup_{R^n}\ \psi(x)\,(u_0(x) - v_0(x))^+ - F(\bar{t}) \ge \frac{\alpha}{2} \tag{3.3}$$

We choose a function $\beta \in C^{\infty}(R \times R^n)$ satisfying

$$0 \le \beta \le 1, \ \ \beta(0,0) = 1, \ \ \beta(t,x) = 0 \ \text{ if } \ |t|^2 + |x|^2 > 1.$$

Let $m = \max_{t\in[0,\tau]} \max_{x\in R^n} \max(\psi(x)|u(t,x)|, \psi(x)|v(t,x)|)$. For $\lambda > 0$ define the function $\Phi : R^n \times R^n \times [0,\tau] \times [0,\tau] \to R$ by

$$\Phi(x,y,t,s) = e^{-\omega t}\psi(x)u(t,x) - e^{-\omega s}\psi(y)v(s,y) - \lambda(t+s) \tag{3.4}$$
$$-\frac{1}{2}(F(t) + F(s)) + M\beta_\epsilon(t-s, x-y)$$

where $M = 5m + 2\lambda\tau + F(\tau)$ and

$$\beta_\epsilon(t,x) = \beta(\frac{t}{\epsilon}, \frac{x}{\epsilon}) \quad \text{for } \ (t,x) \in R \times R^n.$$

Off the support of $\beta_\epsilon(t-s, x-y)$, $\Phi \le 2m$, while if $|x|+|y| \to \infty$ on this support, then $|x|, \ |y| \to \infty$ and thus from (2.3) $\lim_{|x|+|y|\to\infty} \ \Phi \le M$. From (3.3)

$$\Phi(\bar{x},\bar{x},\bar{t},\bar{t}) = e^{-\omega\bar{t}}\psi(\bar{x})(u(\bar{t},\bar{x}) - v(\bar{t},\bar{x})) - F(\bar{t}) - 2\lambda\,\bar{t} + M > M$$

provided that $4\lambda\tau < \alpha$. Thus, if (x_0, y_0, t_0, s_0) attains the maximum of Φ then $x_0, \ y_0 \in R^n$. We next claim that if $\lambda, \ \epsilon > 0$ are sufficiently small then $t_0, \ s_0 \ge \mu$ for some $\mu > 0$ independent of $\lambda, \ \epsilon$. To prove this, we note that

$$\Phi(x,y,t,s) \le 2m \quad \text{if } \ |x-y|^2 + |t-s|^2 \ge \epsilon^2$$

and

$$\sup \ \Phi \ge \sup_{x\in R^n} \Phi(x,x,\tau,\tau) \ge 3m.$$

Thus, $|x_0 - y_0|^2 + |t_0 - s_0|^2 \le \epsilon^2$ and

$$\begin{aligned}\Phi(x_0,y_0,t_0,s_0) \ &\le \ e^{-\omega t_0}\psi(x_0)(u(t_0,x_0) - v(t_0,x_0)) + M + \omega_2(\epsilon) \\ &\le \ \psi(x_0)(u(0,x_0) - v(0,x_0)) + M + \omega_1(t_0) + \omega_2(t_0) + \omega_2(\epsilon)\end{aligned}$$

where $\omega_1(\cdot), \ \omega_2(\cdot)$ are the modulus of continuity of $e^{-\omega t}\psi(x)u(t,x)$ and $e^{-\omega s}\psi(y)v(s,y)$ on $B_r \times [0,\tau]$. Since on the other hand we have from (3.3)

$$\Phi(\bar{x},\bar{x},\bar{t},\bar{t}) \ge \sup_{R^n} \ \psi(u(0,x) - v(0,x)) + \frac{\alpha}{2} + M - 2\lambda\,\bar{t}$$

we have that

$$\omega_1(t_0) + \omega_2(t_0) + \omega_2(\epsilon) \ge \frac{\alpha}{2} - 2\lambda\,\tau.$$

Now, if we choose $\epsilon > 0$ such that $\omega_2(\epsilon) \le \frac{\alpha}{8}$, $\lambda > 0$ such that $2\lambda\tau \le \frac{\alpha}{8}$ and $\mu > 0$ such that $\omega_1(\sigma) + \omega_2(\sigma) \le \frac{\alpha}{4}$ for $0 < \sigma \le \mu$, then we conclude that $t_0 \ge \mu$. Similarly, we obtain $s_0 \ge \mu$ and thus the claim is proved.

Hence Φ attains its maximum value at some point $(x_0, y_0, t_0, s_0) \in R^{2n} \times (0, \tau]^2$. Moreover, $|x_0 - y_0|^2 + |t_0 - s_0|^2 \leq \epsilon^2$. Now (t_0, x_0) is a maximum point of

$$e^{-\omega t}\psi(x)\,(u(t,x) - \phi(t,x))$$

where

$$\phi(t,x) = \frac{e^{-\omega s_0}\psi(y_0)v(s_0,y_0) + \lambda(t+s_0) + \frac{1}{2}(F(t)+F(s_0)) - M\beta_\epsilon(t-s_0, x-y_0) + \Phi(x_0,y_0,t_0,s_0)}{e^{-\omega t}\psi(x)}$$

and since $e^{-\omega \cdot}\psi > 0$ the function $(t,x) \to u(t,x) - \phi(t,x)$ attains a maximum 0 at (t_0, x_0). Since u is a subsolution

$$\omega\, u(t_0,x_0) + \frac{\lambda + \frac{1}{2}F'(t_0) - MD_t\beta_\epsilon(t_0 - x_0, x_0 - y_0)}{e^{-\omega t_0}\psi(x_0)}$$

$$+H(t_0, x_0, u(t_0,x_0), p)) \leq f(t_0, x_0) \tag{3.5}$$

$$\text{with } \; p = (2+\delta)\psi(x_0)u(t_0,x_0)|x_0|^\delta\, x_0 - \frac{MD_x\beta_\epsilon(t_0 - x_0, x_0 - y_0)}{e^{-\omega t_0}\psi(x_0)}.$$

where we used the fact that $\phi(t_0,x_0) = u(t_0,x_0)$. Moreover since $u \in \Sigma_{c_1,c_2}$

$$|p| \leq c_1 + c_2\,|x_0|. \tag{3.6}$$

Similarly, the function

$$(s,y) \to v(s,y) - \frac{e^{-\omega t_0}\psi(x_0)u(t_0,x_0) - \lambda(t_0+s) - \frac{1}{2}(F(t_0)+F(s)) + M\beta_\epsilon(t_0 - s, x_0 - y) - \Phi(x_0,y_0,t_0,s_0)}{e^{-\omega s}\psi(y)}$$

attains a minimum 0 at (s_0, y_0) and since v is a super solution

$$\omega\, v(s_0,y_0) - \frac{\lambda + \frac{1}{2}F'(s_0) + MD_t\beta_\epsilon(t_0 - s_0, x_0 - y_0)}{e^{-\omega s_0}\psi(y_0)}$$

$$+H(s_0, y_0, v(s_0,y_0), q)) \geq g(s_0, y_0) \tag{3.7}$$

$$\text{with } \; q = (2+\delta)\psi(s_0,y_0)v(s_0,y_0)|y_0|^\delta\, y_0 - \frac{MD_x\beta_\epsilon(t_0 - s_0, x_0 - y_0)}{e^{-\omega s_0}\psi(y_0)}.$$

where $|q| \leq c_1 + c_2\,|x_0|$. Thus by (3.5) and (3.7) we have

$$\omega\,(e^{-\omega t_0}\psi(x_0)u(t_0,x_0) - e^{-\omega s_0}\psi(s_0,y_0)v(s_0,y_0)) + 2\lambda + \tfrac{1}{2}\,(F'(t_0) + F'(s_0))$$

$$\leq e^{-\omega s_0}\psi(y_0)H(s_0,y_0,v(s_0,y_0),q) - e^{-\omega t_0}\psi(x_0)H(t_0,x_0,u(t_0,x_0),p) \tag{3.8}$$

$$+e^{-\omega t_0}\psi(x_0)f(t_0,x_0) - e^{-\omega s_0}\psi(y_0)g(s_0,y_0).$$

Since $\Phi(x_0, y_0, t_0, s_0) \geq \Phi(\bar{x}, \bar{x}, \bar{t}, \bar{t})$

$$(e^{-\omega t_0}\psi(x_0) - e^{-\omega s_0}\psi(y_0))\, u(t_0, x_0) + e^{-\omega s_0}\psi(y_0)(u(t_0, x_0) - v(s_0, y_0))$$

$$\geq e^{-\omega\bar{t}}\psi(\bar{x})(u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{x})) + M\,(1 - \beta_\epsilon(t_0 - s_0, x_0 - y_0)) - F(\bar{t}) - 2\lambda\,\bar{t},$$

it thus follows from (2.11) and (3.3) that $u(t_0, x_0) \geq v(s_0, y_0)$ for sufficiently small $\epsilon > 0,\ \lambda > 0$. From (3.1) and (3.8) and by the arguments leading to the estimate (2.12), we have

$$\begin{aligned}
&\omega\,(e^{-\omega t_0}\psi(x_0)u(t_0, x_0) - e^{-\omega s_0}\psi(y_0)v(s_0, y_0))\\
&\quad +2\lambda + \frac{1}{2}\,(F'(t_0) + F'(s_0)) - (e^{-\omega t_0}\psi(x_0)f(t_0, x_0) - e^{-\omega s_0}\psi(y_0)g(s_0, y_0))\\
&\quad \leq O(\epsilon) + a_0(e^{-\omega t_0}\psi(x_0)u(t_0, x_0) - e^{-\omega s_0}\psi(y_0)v(s_0, y_0))\\
&\quad\quad +e^{-\omega s_0}\psi(y_0)((a(y_0), p - q) + a_2\,(1 + c_1 + c_2 r)|p - q|)
\end{aligned} \tag{3.9}$$

where $r = \max\{|x_0|, |y_0|\}$. Hence using exactly the same arguments as those in the proof of Theorem 2.1, it follows from the expression of p, q in (3.5) and (3.7), respectively that for any $\epsilon > 0$ there exists a constant $\bar{c} > 0$ (depends on ε and $c_1 a_2 + a_3$) such that for $c \geq \bar{c}$

$$e^{-\omega s_0}\psi(y_0)((a(y_0), p - q) + a_2\,(1 + c_1 + c_2 r)|p - q|)$$

$$\leq O(\epsilon) + ((2 + \delta)\max(0, a_1 + c_2 a_2) + \varepsilon)\,(e^{-\omega t_0}\psi(x_0)u(t_0, x_0) - e^{-\omega s_0}\psi(y_0)v(s_0, y_0)).$$

Now, from (3.9) we obtain

$$\begin{aligned}
&2\lambda \leq O(\epsilon) + e^{-\omega t_0}\psi(x_0)f(t_0, x_0) - e^{-\omega s_0}\psi(y_0)g(t_0, x_0)\\
&\quad -\frac{1}{2}\,(e^{-\omega t_0}\sup_{R^n}\ \psi(f(t_0, \cdot)) - g(t_0, \cdot))^+ + e^{-\omega s_0}\sup_{R^n}\ \psi(f(s_0, \cdot)) - g(s_0, \cdot))^+)
\end{aligned} \tag{3.10}$$

where we chose $\omega = a_0 + (2 + \delta)\,\max(0, a_1 + c_2 a_2) + \varepsilon$. By letting $\epsilon \to 0$ in (3.10) we obtain $\lambda \leq 0$ which contradicts the assumption. Thus, the assumption (3.2) is false and therefore

$$e^{-\omega t}\sup_{R^n}\ \psi(x)(u(t, x) - v(t, x))^+ \leq \sup_{R^n}\ \psi(x)(u_0(x) - v_0(x))^+ + F(t)$$

on $[0, \tau]$. □

Letting $\delta \to 0^+$ we obtain the following theorem.

Theorem 3.2 *Assume that $u,\ v \in \Sigma_{c_1,c_2}$, u is a subsolution of $u_t + H(t, x, u, u_x) = f$ and v is a supersolution of $v_t + H(t, x, v, v_x) = g$. Then, for any $\varepsilon > 0$ there exists a constant $\bar{c} > 0$ (depends on ε and $c_1 a_2 + a_3$) such that for $c \geq \bar{c}$*

$$\sup_{R^n}\ \frac{(u(t, x) - v(t, x))^+}{c + |x|^2} \leq e^{\omega t}\sup_{R^n}\ \frac{(u_0 - v_0)^+}{c + |x|^2} + \int_0^t e^{\omega(t-s)}\sup_{R^n}\ \frac{(f(s, x) - g(s, x))^+}{c + |x|^2}\,ds$$

where $\omega = a_0 + 2\,\max(0, a_1 + c_2 a_2) + \varepsilon$.

If u , $v \in \Sigma_{c_1,c_2}$ are viscosity solutions to $u_t + H(t,x,u,u_x) = f$ and $v_t + H(t,x,v,v_x) = g$, respectively, then it follows from Theorem 3.2 that

$$\sup_{R^n} \frac{|u(t,x) - v(t,x)|}{c + |x|^2} \le e^{\omega t} \sup_{R^n} \frac{|u_0 - v_0|}{c + |x|^2} + \int_0^t e^{\omega(t-s)} \sup_{R^n} \frac{|f(s,x) - g(s,x)|}{c + |x|^2}\, ds.$$

This implies the uniqueness of viscosity solutions to (1.3) in the class Σ_{c_1,c_2}. Moreover, we have the following corollary.

Corollary 3.3 *Assume that* u, $v \in \Sigma_{c_1,c_2}$, *u is a subsolution of* $u_t + H(t,x,u,u_x) = f$ *and v is a supersolution of* $v_t + H(t,x,v,v_x) = g$. *Suppose that* $c_1 = 0$ *and* $a_3 = 0$ *in (3.1) and the functions*

$$\frac{u(t,x)}{|x|^2}, \quad \frac{v(t,x)}{|x|^2}, \quad \frac{f(t,x)}{|x|^2}, \quad \textit{and} \quad \frac{g(t,x)}{|x|^2}$$

are continuous at $x = 0$ *(consequently,* $u(0) = v(0) = f(0) = g(0) = 0$*). Then, we have*

$$\sup_{R^n} \frac{(u(t,x) - v(t,x))^+}{|x|^2} \le e^{\omega t} \sup_{R^n} \frac{(u_0 - v_0)^+}{|x|^2} + \int_0^t e^{\omega(t-s)} \sup_{R^n} \frac{(f(s,x) - g(s,x))^+}{|x|^2}\, ds$$

where $\omega = a_0 + 2\,(a_1 + c_2 a_2)$.

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