Linear Algebra lecture notes

Kazufumi Ito*

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1 Introduction

Probably the most important problem in mathematics is that of solving a system of linear equations. Well over 75 percent of all mathematical problems encountered in scientific or industrial applications involve solving a linear system at some stage. By using the methods of modern mathematics, it is often possible to take a sophisticated problem and reduce it to a single system of linear equations. Linear systems arise in applications to such areas

*Department of Mathematics, North Carolina State University, Raleigh, North Carolina, USA

as business, economics, sociology, ecology, demography, genetics, electronics, engineering, physics, statistics, neuron-network and AI. Therefore, it seems appropriate to begin the lecture with a section on linear systems.

A linear map A maps a column vector x of dimension n into a vector y of dimension m by $(A(\alpha x_1 + \beta x_2) = \alpha\, Ax_1 + \beta\, Ax_2)$.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

onto a vector y of dimension m by

$$\mathbf{y} = A(\mathbf{x}) = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}.$$

The linear map A is thus defined by the matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

and maps the column vector $\mathbf{x}$ to the matrix product

$$\mathbf{y} = \mathbf{A}\mathbf{x}.$$

A Linear system of equations is that $Ax = y$ and we look for a solution x (vector) given right had-side vector y.

The dot product of a vector $a = (a_1, a_2 \cdots a_n)$ with x is defined as

$$a \cdot x = a_1x_1 + a_2x_2 + \cdots a_nx_n,$$

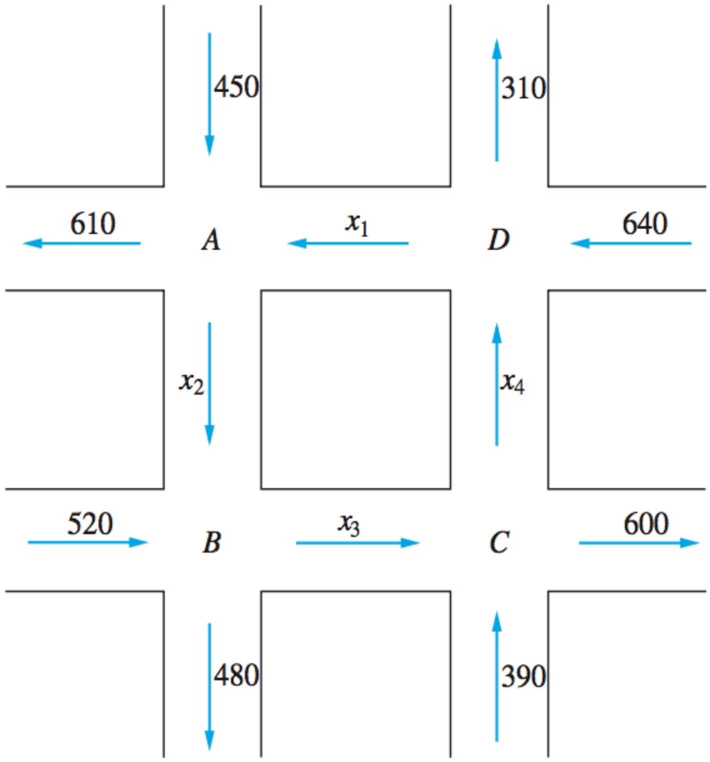
i.e., by multiplying term-by-term the entries of a and x and summing these n products Then, y_i is the dot product of the i-th row of A and x.

EXAMPLE (**traffic flow**)

In the downtown section of a certain city, two sets of one-way streets intersect as shown in Figure. The average hourly volume of traffic entering and leaving this section during rush hour is given in the diagram.

At each intersection the number of automobiles entering must be the same as the number leaving

$$\begin{aligned} x_1 + 450 &= x_2 + 610 \quad \text{(intersection A)} \\ x_2 + 520 &= x_3 + 480 \quad \text{(intersection B)} \\ x_3 + 390 &= x_4 + 600 \quad \text{(intersection C)} \\ x_4 + 640 &= x_1 + 310 \quad \text{(intersection D)} \end{aligned}$$



Thus, we obtain a system of linear equations:

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 160 \\ -40 \\ 210 \\ -330 \end{pmatrix}.$$

A defines a linear map for R^n into R^m, i.e.,

$$A(a_1\, x_1 + a_2\, x_2) = a_1\, Ax_1 + a_2\, Ax_2$$

for all $a_1, a_2 \in R$ and vectors $x_1,\ x_2$.

EXAMPLE 2

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 5 & 3 & 7 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} 4x_1 + 2x_2 + x_3 \\ 5x_1 + 3x_2 + 7x_3 \end{bmatrix}$$

■

EXAMPLE 3

$$A = \begin{bmatrix} -3 & 1 \\ 2 & 5 \\ 4 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} -3 \cdot 2 + 1 \cdot 4 \\ 2 \cdot 2 + 5 \cdot 4 \\ 4 \cdot 2 + 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 24 \\ 16 \end{bmatrix}$$

■

EXAMPLE 4 Write the following system of equations as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 5 \\ x_1 - 2x_2 + 5x_3 &= -2 \\ 2x_1 + x_2 - 3x_3 &= 1 \end{aligned}$$

Solution

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -2 & 5 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

■

Matrix product $C = AB$ In general If A is an $m \times n$ matrix and B is an $n \times p$ matrix (Note: number of columns of A and number of rows of B must be the same n).

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

the matrix product $C = AB$ is defined to be the $m \times p$ matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

such that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj},$$

for $i = 1, \cdots, m$ and $j = 1, \cdots, p$. That is, the entry c_{ij} is the dot product of the i th row of A and the jth column of B.

Therefore, $C = AB$ can also be written as

$$\mathbf{C} = \begin{pmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + \cdots + a_{1n}b_{n2} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + \cdots + a_{2n}b_{n2} & \cdots & a_{21}b_{1p} + \cdots + a_{2n}b_{np} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & a_{m1}b_{12} + \cdots + a_{mn}b_{n2} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{pmatrix}.$$

Thus the product AB is defined if and only if the number of columns in A equals the number of rows in B, in this case n.

Note: An element s belongs to a set $S \Leftrightarrow s \in S$.

$A \subset B$**: a set** A **is a subset of a set** B, or equivalently B is a superset of A, if A is contained in B. That is, all elements of A are also elements of B.

For example, Q is subset of R and R is a subset of C.

Matrices make sense over more than just the real numbers.

We recall some of the most common alternative choices:

1. The natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.

   In $\mathbb{N}$, addition is possible but not subtraction; e.g. $2 - 3 \notin \mathbb{N}$.

2. The integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$.

   In $\mathbb{Z}$, addition, subtraction and multiplication are always possible, but not division; e.g. $2/3 \notin \mathbb{Z}$.

3. The rational numbers $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z},\ q \neq 0\}$.

   In $\mathbb{Q}$, addition, subtraction, multiplication and division (except by zero) are all possible. However, $\sqrt{2} \notin \mathbb{Q}$.

4. The real numbers $\mathbb{R}$. These are the numbers which can be expressed as decimals. The rational numbers are those with finite or recurring decimals.

   In $\mathbb{R}$, addition, subtraction, multiplication and division (except by zero) are still possible, and all positive numbers have square roots, but $\sqrt{-1} \notin \mathbb{R}$.

5. The complex numbers $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$, where $i^2 = -1$.

   In $\mathbb{C}$, addition, subtraction, multiplication and division (except by zero) are still possible, and all, numbers have square roots. In fact all polynomial equations with coefficients in $\mathbb{C}$ have solutions in $\mathbb{C}$.

While matrices make sense in all of these cases, we will focus on the last three, where there is a notion of addition, multiplication, and division.

2 Vector space

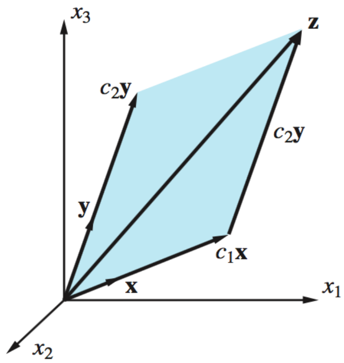
Linear algebra is the study of linear maps on finite-dimensional vector spaces. Eventually we will learn what all these terms mean. In this chapter we will define vector spaces and discuss their elementary properties. We recall an n-tuple of real numbers as a column vector

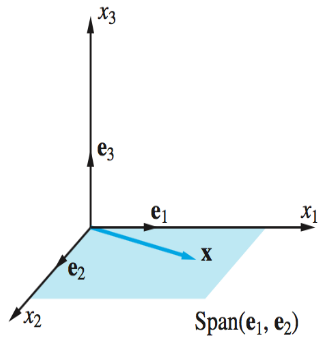
$$x = \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

For example, the solution of the linear system. A vector space is a collection of vectors. The operations of addition and scalar multiplication rules for vectors are

$$a_1 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + a_2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a_1u_1 + a_2v_1 \\ a_1u_2 + a_2v_2 \\ a_1u_3 + a_2v_3 \end{pmatrix}$$

for $a_1,\ a_2 \in R$ and vectors $\vec{u}, \vec{v} \in R^3$.





LEARNING OBJECTIVES FOR THIS CHAPTER:

- Vector space and subspace
- Linear independent vectors and span
- Gauss elimination (method to solve $Ax = b$).
- Bases
- Dimension of subspace

Next, we present the formal definition of a vector space.

Definition (Field F) A field is a set F together with two binary operations on F called addition and multiplication. A binary operation on F is a mapping $F \times F \to F$, that is, a correspondence that associates with each ordered pair of elements of F a uniquely determined element of F. The result of the addition of a and b is called the sum of a and b, and is denoted $a + b$. Similarly, the result of the multiplication of a and b is called the product of a and b, and is denoted ab or $a \cdot b$. These operations are required to satisfy the following properties, referred to as field axioms. In these axioms, a, b, and c are arbitrary elements of the field F.

- Associativity of addition and multiplication: $a+(b+c) = (a+b)+c$, and $a(bc) = (ab)c$.
- Commutativity of addition and multiplication: $a + b = b + a$, and $ab = ba$.
- Additive and multiplicative identity: there exist two different elements 0 and 1 in F such that $a + 0 = a$ and $a1 = a$.
- Additive inverses: for every a in F, there exists an element in F, denoted $-a$, called the additive inverse of a, such that $a + (-a) = 0$.
- Multiplicative inverses: for every $a \neq 0$ in F, there exists an element in F, denoted by a^{-1} or $1/a$, called the multiplicative inverse of a, such that $aa^{-1} = 1$.
- Distributivity of multiplication over addition: $a(b + c) = (ab) + (ac)$.

This may be summarized by saying: a field has two operations, called addition and multiplication; it is an abelian group under addition with 0 as the additive identity; the nonzero elements are an abelian group under multiplication with 1 as the multiplicative identity; and multiplication distributes over addition.

N, Z are not field. C, R and Q are all fields. There are many other fields, including some finite fields. For example, for each prime number p, there is a field F_p = {0,1,2,...,p-1} with p elements, where addition and multiplication are carried out modulo p. Thus, in F_7, we have $5+4=2$, $5 \times 4 = 6$ and $5^{-1} = 3$ because $5 \times 3 = 1$. The smallest such field F_2 has just two elements 0 and 1, where $1+1=0$. This field is extremely important in Computer Science since an element of F_2 represents a bit of information.

Definition (Vector Space V) $v = \vec{v} \in V$

- Associativity of addition: $u + (v + w) = (u + v) + w$
- Commutativity of addition: $u + v = v + u$
- Identity element of addition: There exists an element $0 \in V$, called the zero vector, such that $v + 0 = v$ for all $v \in V$.
- Inverse elements of addition: For every $v \in V$, there exists an element $-v \in V$ called the additive inverse of v, such that $v + (-v) = 0$.
- Compatibility of scalar multiplication with field multiplication: $a(bv) = (ab)v$.
- Identity element of scalar multiplication: $1v = v$, where 1 denotes the multiplicative identity in F.
- Distributivity of scalar multiplication with respect to vector addition: $a(u + v) = au + av$.
- Distributivity of scalar multiplication with respect to field addition: $(a+b)v = av+bv$

EXAMPLEs (1) The most familiar examples are

$$R^2 = \{\vec{x} = (x_1, x_2),\ x_1, x_2 \in R\} \text{ and } R^3 = \{\vec{x} = (x_1, x_2, x_3),\ x_1, x_2.x_3 \in R\}$$

which we can think of geometrically as the points in ordinary 2 and 3-dimensional space, equipped with a coordinate system. In general

$$R^n = \{\vec{x} = (x_1, x_2, \cdots, x_n),\ x_1, x_2, \cdots, x_n \in R\}$$

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$$

$$c\,\vec{u} = \begin{pmatrix} c\,u_1 \\ c\,u_2 \\ c\,u_3 \end{pmatrix}$$

$$(2\vec{u}+\vec{v})-3\vec{w}=\begin{pmatrix} 2\,u_1 \\ 2\,u_2 \\ 2\,u_3 \end{pmatrix}+\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}-\begin{pmatrix} 3w_1 \\ 3w_2 \\ 3w_3 \end{pmatrix}=\begin{pmatrix} 2u_1+v_1-3w_1 \\ 2u_2+v_2-3w_2 \\ 2u_3+v_3-3w_3 \end{pmatrix}$$

(2) The set $R^{m\times n}$ of all $m\times n$ matrices is itself a vector space over R using the operations of addition and scalar multiplication.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

$$c\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}=\begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \\ ca_{31} & ca_{32} & ca_{33} \end{pmatrix}$$

(3) Let P_n be the set of polynomials in x with coefficients in the field F. That is,

$$P_n=\{a_0+a_1x+\cdots+a_nx^n,\ \ a_i\in R\}.$$

Let $C((0,1),R)$, consisting of all functions $f:(0,1)\to R$ with the usual pointwise definitions of addition and scalar multiplication of functions.

$$(f+g)(t)=f(t)+g(t),\quad (cf)(t)=cf(t)\text{ for all } t\in[0,1].$$

We shall assume the following additional simple properties of vectors and scalars from now on. They can all be deduced from the axioms (and it is a useful exercise to do so).

$$(i)a\vec{0}=\vec{0},\ (ii)0\vec{v}=\vec{0},\ (iii)-(av)=(-a)v=a(-v),\ (iv)a\vec{v}=\vec{0}\to a=0\text{ or }\vec{v}=0.$$

for all $a\in F$ and $v\in V$.

(1) Identity element of addition 0 is unique. Proof: $0=0+0'=0'$

(2) The additive inverse is unique. $u+v=0=u+v'$ implies $v=v'$. Proof:

$$v=v+0=v+u+v'=v'+(u+v)=v'+0=v'$$

(ii) Proof: $0v=(0+0)v=0v+0v$ and $0v=0v=0v+w$ for all $w\in V$. Thus, $0v=\vec{0}$. Convention: $\vec{u}-\vec{v}=\vec{u}+(-1)\vec{v}=\vec{u}+(-\vec{v})$

2.1 Subspaces

Definition (subspace) A subset U of vector space V is called a subspace of V if U is also a vector space (using the same addition and scalar multiplication as on V).

EXAMPLEs

$$\left\{\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}:\ x,y\in R\right\},\quad \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}: x+2y-z=0\right\}$$

are subspace of R^3

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + 2y - z = 1 \right\}$$

is not a subspace of R^3.

$$\left\{ \begin{pmatrix} x & y \\ y & z \end{pmatrix} : x, y, z \in R \right\} \text{ (symmetric matrix)}$$

is a subspace in $R^{2\times 2}$
(2) The null space of a matrix A

$$N(A) = \{x \in R^h : Ax = \vec{0}\}$$

is a subspace of R^n since

$$A(a_1\vec{x}_1 + a_2\vec{x}_2) = a_1 A\vec{x}_1 + a_2 A\vec{x}_2.$$

The range space of a matrix A

$$R(A) = \{y \in R^m : y = Ax,\ x \in R^n\}$$

is a subspace of R^m since

$$a_1 y_1 + a_2 y_2 = A(a_1 x_1 + a_2 x_2) \text{ for } y_1 = Ax_1,\ y_2 = Ax_2.$$

(3) P_2 is a subspace of P_3
(4) The space of all continuously differentiable function $C^1(0,1)$ on $(0,1)$ is a subspace of the space of continuous functions $C(0,1)$
(5) $\{f'(1/2) = f(1/2)\}$ is a subspace of $C^1(0,1)$.

(6) Let S be the set of all $f \in C^2(0,1)$ such that $f'' + f(x) = 0$. is a subspace of $C^2(0,1)$. In fact

$$(a_1 f + a_2 g)'' + (a_1 f + a_2 g) = a_1(f'' + f(x)) + a_2(g'' + g(x)) = 0$$

and $a_1 f + a_2 g \in S$ for all $f,\ g \in S$.

Proposition If W_1 and W_2 are subspaces of V then so is $W_1 \cap W_2$.
Proof. Let $u, v \in W_1 \cap W_2$ and $a \in F$. Then $u + v \in W_1$ (because W_1 is a subspace) and $u + v \in W_2$ (because W_2 is a subspace). Hence $u + v \in W_1 \cap W_2$. Similarly, we get $av \in W_1 \cap W_2$, so $W_1 \cap W_2$ is a subspace of V.

Warning! It is not necessarily true that $W_1 \cup W_2$ is a subspace, as the following example shows.
EXAMPLE Let $V = R^2$, let $W_1 = \{(a, 0) : a \in R\}$ and $W_2 = \{(0, b) : b \in R\}$. Then $W_1,\ W_2$ are subspaces of V , but $W_1 \cup W_2$ i s not a subspace, because $(1, 0), (0, 1) \in W_1 \cup W_2$, but $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$.

Note that any subspace of V that contains W_1 and W_2 has to contain all vectors of the form $u + v$ for $u \in W_1$, $v \in W_2$. This motivates the following definition.

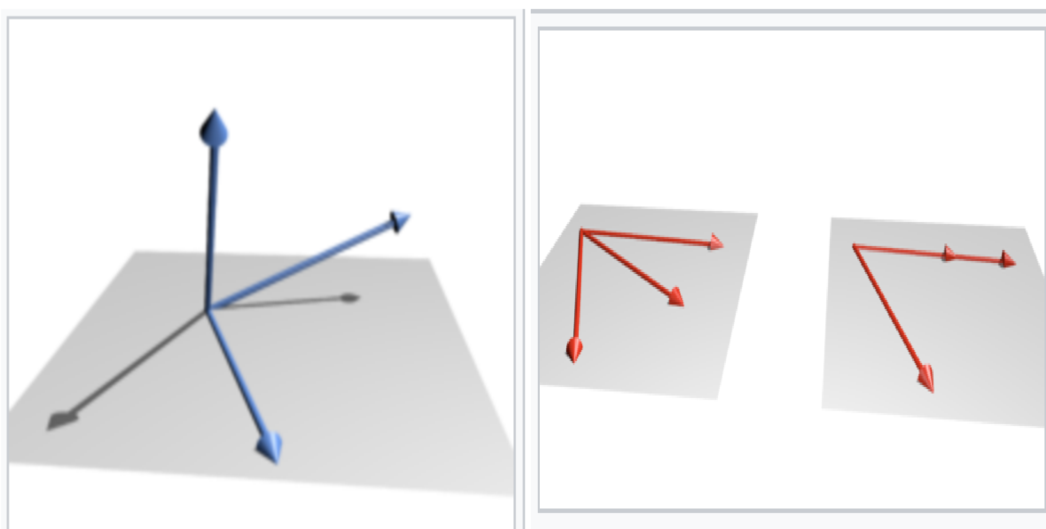
Definition Let W_1, W_2 be subspaces of the vector space V . Then the direct sum of W_1, W_2 is

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

Do not confuse $W_1 + W_2$ with $W_1 \cup W_2$.

Proposition If W_1, W_2 are subspaces of V then so is $W_1 + W_2$. In fact, it is the smallest subspace that contains both W_1 and W_2.
Proof. Let $u, v \in W_1 + W_2$. Then $u = u_1 + u_2$ for some $u_1 \in W_1, u_2 \in W_2$ and $v = v_1 + v_2$ for some $v_1 \in W_1, v_2 \in W_2$. Then $u + v = (u_1 + v_1) + (u_2 + v_2) \in W_1 + W_2$. Similarly, if $a \in F$ then $av = av_1 + av_2 \in W_1 + W_2$. Thus $W_1 + W_2$ is a subspace of V. Any subspace of V that contains both W_1 and W_2 must contain $W_1 + W_2$, so it is the smallest such subspace.

2.2 Linear independent



Definition A sequence of vectors $(\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k})$from a vector space V is said to be linearly dependent, if there exist scalars $a_1, a_2, \ldots, a_k$, not all zero, such that

$$a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_k\vec{v_k} = \vec{0}.$$

Notice that if not all of the scalars are zero, then at least one is non-zero, say a_1, in which case this equation can be written in the form

$$\vec{v_1} = \frac{-a_2}{a_1}\vec{v_2} + \cdots + \frac{-a_k}{a_1}\vec{v_k}.$$

Thus, $\vec{v_1}$ is shown to be a linear combination of the remaining vectors.

A sequence of vectors $(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n})$ is said to be linearly independent if the equation

$$a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n} = \vec{0},$$

can only be satisfied by $a_i = 0, i = 1, \cdots, n$. This implies that no vector in the sequence can be represented as a linear combination of the remaining vectors in the sequence. Even more concisely, a sequence of vectors is linear independent if and only if $\vec{0}$ can be represented as a linear combination of its vectors in a unique way.

The alternative definition, that a sequence of vectors is linearly dependent if and only if some vector in that sequence can be written as a linear combination of the other vectors.

Remark: (1) Let $\vec{v}_i,\ 1 \le i \le n$ be column vectors

$$\vec{v}_i = \begin{pmatrix} v_{1,i} \\ \vdots \\ v_{m,i} \end{pmatrix} \in R^m$$

Then,

$$\begin{pmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & & \vdots \\ v_{m,1} & \cdots & v_{m,n} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(2) $\{\vec{v}_k\}$ are linearly independent $Ax = \vec{0}$ has a unique solution $x = \vec{0}$ and $N(A) = \{\vec{0}\}$ (null space of A). Moreover $Ax = b$ has a unique solution, i.e., $Ax_1 = b$ and $Ax_2 = b$ implies $A(x_1 - x_2) = \vec{0}$ and thus $x_1 = x_2$.

(3) $\{\vec{v}_k\}$ are linearly dependent $Ax = \vec{0}$ has a nontrivial solution.

Question 1 and Objective: Identify linearly independent or dependent. How to find $N(A)$ and $R(A)$.

2.3 Span

Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ be vectors in a vector space V. A sum of the form

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

where $a_1, \cdots a_n \in R$, is called a linear combination of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ is called the span of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$, i.e.,

$$Span(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n,\ a_i \in R\}.$$

which is a subspace of V.

Remark Let $\vec{e}_i$ is the $i-$th unit vector such that $(\vec{e}_i)_j = 0,\ j \ne i$ and $(\vec{e}_i)_i = 1$ Then, $\{\vec{e}_i\}_{i=1}^n$ are linear independent and $R^n = span(\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n)$, i.e.,

$$\vec{x} = (x_1, \cdots, x_n) = x_1\,\vec{e}_1 + x_2\,\vec{e}_2 \cdots + x_n\vec{e}_n.$$

$$span(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n) = V_1 + V_2 + \cdots V_n \ \ \text{(direct sum)}$$

where $V_i = span(\vec{v}_i)$ Recall that if $(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n)$ are linear dependent, i.e. $a_1 \ne 0$, then $\vec{v}_1 \in span(\vec{v}_2, \cdots, \vec{v}_n))$ and thus $span(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n) = span(\vec{v}_2, \cdots, \vec{v}_n))$.

$$span(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n) = R^n, \ \ \vec{v}_i \in R^n$$

if and only if $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ are linearly independent.

Question 2 and Objective: How to determine the span of vectors.

EXAMPLEs (1) $P_3 = span(1, x, x^2, x^3)$

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

(2) Vectors $\{1, x, x^2, x^3\}$ are linearly independent in P_3. Suppose x^3 are linearly dependent, i.e.

$$x^3 = a_0 + a_1 x + a_2 x^2 \text{ for some } a_0,\ a_1,\ a_2 \in R$$

Taking derivative of this three times in x, we obtain $6 = 0$, which is a contradiction.

(3) **Vectors** $\{a_{11}+a_{12}x+a_{13}x^2, a_{21}+a_{22}x+a_{23}x^2, a_{31}+a_{32}x+a_{33}x^2\}$ are linearly independent if and only if $N(A) = \{\vec{0}\}$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

(4) the sum of two vectors $\vec{v}_1$ and $\vec{v}_2$ is the plane that contains them both. For $\vec{v}_1 = (1, 2, 3)^t, \vec{v}_2 = (2, 4, 6)^t$ are linearly dependent and

$$span(\vec{v}_1, \vec{v}_2) = span(\vec{v}_1).$$

(5) **If** $span(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n) = R^m$**, then** $n \geq m$**. Conversely, if** $n > m$**, then** $(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n)$ **are linear dependent.**

2.4 Gauss-Jordan Reduction

LEARNING OBJECTIVES FOR THIS SECTION: Gauss elimination to Triangular matrix form U, and LU decomposition of matrix A. Examples and Applications.

In this section we study a way to solve a linear equation $Ax = b$. If there exits a unique solution to $Ax = b$ we say

$$x = A^{-1}b,$$

where $A^{-1} \in R^{n \times n}$ is the inverse of a matrix A, i.e., $A^{-1}A = I$ =identity matrix.

Objective includes: Identify $\{\vec{v}_1, \cdots, \vec{v}_n\}$ linearly independent or dependent. Find $N(A)$ of matrix A of column vectors $\{\vec{v}_1, \cdots, \vec{v}_n\}$:

$$A = [\vec{v}_1 | \vec{v}_2 | \cdots | \vec{v}_n].$$

EXAMPLE 1 (**Triangular System**)

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 1 \\ x_2 - x_3 = 2 \\ 2x_3 = 4 \end{cases} \Leftrightarrow \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

is in upper triangular form, since in matrix A has all zeroes under the diagonal. Because of the strict triangular form, the system is easy to solve. It follows from the third equation that $x_3 = 2$. Using this value in the second equation, we obtain

$$x_2 - 2 = 2 \Rightarrow x_2 = 4$$

Using $x_2 = 4, x_3 = 2$ in the first equation, we end up with

$$3x_1 + 2 \cdot 4 + 2 = 1 \Rightarrow x_1 = -3$$

Thus, the solution of the system is $(-3, 4, 2)$.

Any $n \times n$ upper triangular system can be solved in the same manner as the last example. First, the nth equation is solved for the value of x_n. This value is used in the $(n-1)$st equation to solve for x_{n-1}. The values x_n and x_{n-1} are used in the $(n-2)$nd equation to solve for x_{n-2}, and so on. We will refer to this method of solving a upper triangular system as **back substitution**.

Remark **If all diagonal entries of upper triangle matrix A are nonzero, then $Ax = b$ has a unique solution by back substitution.** $Ax = \vec{0}$ has a unique solution $x = \vec{0}$, equivalently $N(A) = \{\vec{0}\}$ and $\{\vec{v}_1, \cdots, \vec{v}_n\}$ are linearly independent.

Gauss-Jordan Reduction is to transform A into a upper triangular matrix by row operations as below (Gauss elimination).

EXAMPLE 2 Solve the system

$$\begin{cases} x_1 + 2x_2 + x_3 = 3 \\ 3x_1 - x_2 - 3x_3 = -1 \\ 2x_1 + 3x_2 + x_3 = 4 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}.$$

Subtracting 3 times the first row from the second row yields $-7x_2 - 6x_3 = -10$.
Subtracting 2 times the first row from the third row yields $-x_2 - x_3 = -2$.
If the second and third equations of our system, respectively, are replaced by these new equations, we obtain the equivalent system

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -7 & -6 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -10 \\ -2 \end{pmatrix}$$

If the third equation of this system is replaced by the sum of the third equation and $-\frac{1}{7}$ times the second equation, we end up with the following upper triangular system:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -7 & -6 \\ 0 & 0 & -\frac{1}{7} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -10 \\ -\frac{4}{7} \end{pmatrix}$$

Using back substitution, we get $x_3 = 4, x_2 = -2, x_1 = 3$.

With each system of equations $Ax = b$ we may associate an augmented matrix of the form

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{1n} & b_m \end{bmatrix}.$$

where we attach to the coefficient matrix A an additional column b. The system can be solved by performing operations on the augmented matrix. The xi?s are placeholders that can be omitted until the end of the computation. Corresponding to the three operations used to obtain equivalent systems, the following row operations may be applied to the augmented matrix:

Elementary Row Operations

- [I] Interchange two rows.
- [II] Multiply a row by a nonzero real number.
- [III] Replace a row by its sum with a multiple of another row.

EXAMPLE 3 Solve the system

$$\begin{cases} 0x_1 - x_2 - x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 6 \\ 2x_1 + 4x_2 + x_3 - 2x_4 = -1 \\ 3x_1 + x_2 - 2x_3 + 2x_4 = 3 \end{cases}$$

The augmented matrix for this system is

$$\begin{bmatrix} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{bmatrix}.$$

Since it is not possible to eliminate any entries by using 0 as a **pivot element**, we will use row operation [I] to interchange the first two rows of the augmented matrix. The new first row will be the pivotal row and the pivot element will be 1:

$$(\text{pivot } a_{11 \neq 0}) \to \begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{bmatrix}.$$

Row operation [III] is then used twice to eliminate the two nonzero entries in the first column:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & -4 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{bmatrix}.$$

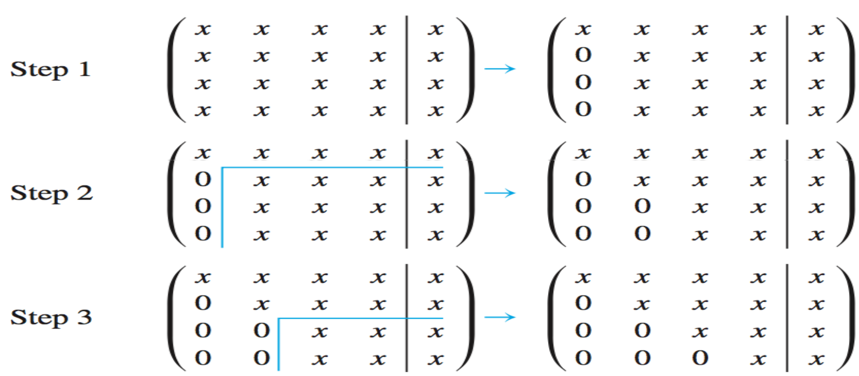
Next, the second row is used as the pivotal row to eliminate the entries in the second column below the pivot element -1:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & -3 & -3 & -15 \end{bmatrix}.$$

Finally, the third row is used as the pivotal row to eliminate the last element in the third column:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}.$$

This augmented matrix represents a upper triangular system. Solving by back substitution, we obtain the solution $(2, -1, 3, 2)$. In general, if an $n \times n$ linear system can be reduced to upper triangular form, then it will have a unique solution that can be obtained by performing back substitution on the triangular system. We can think of the reduction process as an algorithm involving $n-1$ steps. At the first step, a pivot element is chosen from among the nonzero entries in the first column of the matrix. The row containing the pivot element is called the pivotal row. We interchange rows (if necessary) so that the pivotal row is the new first row. Multiples of the pivotal row are then subtracted from each of the remaining $n-1$ rows so as to obtain 0s in the first entries of rows 2 through n. At the second step, a pivot element is chosen from the nonzero entries in column 2, rows 2 through n, of the matrix. The row containing the pivot is then interchanged with the second row of the matrix and is used as the new pivotal row. Multiples of the pivotal row are then subtracted from the remaining $n-2$ rows so as to eliminate all entries below the pivot in the second column. The same procedure is repeated for columns 3 through $n-1$. Note that at the second step row 1 and column 1 remain unchanged, at the third step the first two rows and first two columns remain unchanged, and so on. At each step, the overall dimensions of the system are effectively reduced by 1 If the elimination process can be carried out as described, we will arrive at an equivalent strictly triangular system after $n-1$ steps. The steps of Gauss elimination is depicted by



However, the procedure will break down if, at any step, all possible choices for a pivot element equals to 0. When this happens, the alternative is to reduce the system to certain special echelon, or staircase-shaped, forms. These echelon forms will be studied in the next section. They will also be used for $m \times n$ systems, where $m \neq n$.

2.5 Reduced Row Echelon Form

Gauss-Jordon procedure will break down if, at any step, all possible choices for a pivot element are equal to 0. When this happens, the alternative is to reduce the system to

certain special echelon, or staircase-shaped, forms. A matrix is in row echelon form if it has the shape resulting from a Gaussian elimination. Specifically, a matrix is in row echelon form if
(a) all rows consisting of only zeroes are at the bottom.
(b) the leading coefficient (also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

These two conditions imply
(c) all entries in a column below a leading coefficient are zeros.

Consider the system represented by the augmented matrix

$$\left(\begin{array}{ccccc|c} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right) \begin{array}{l} \leftarrow \text{pivotal row} \\ \\ \\ \\ \\ \end{array}$$

If row operation III is used to eliminate the nonzero entries in the last four rows of the first column, the resulting matrix will be

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{0} \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right) \begin{array}{l} \\ \leftarrow \text{pivotal row} \\ \\ \\ \\ \end{array}$$

At this stage, the reduction to strict triangular form breaks down. All four possible choices for the pivot element in the second column are 0. How do we proceed from here? Since our goal is to simplify the system as much as possible, it seems natural to move over to the third column and eliminate the last three entries:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{3} \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

In the fourth column, all the choices for a pivot element are 0; so again we move on to the next column. If we use the third row as the pivotal row, the last two entries in the fifth column are eliminated and we end up with the matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right]$$

The coefficient matrix that we end up with is not in strict triangular form; it is in staircase, or echelon, form. The horizontal and vertical line segments in the array for the coefficient matrix indicate the structure of the staircase form. Note that the vertical drop is 1 for each step, but the horizontal span for a step can be more than 1.

The equations represented by the last two rows are

$$\begin{aligned} 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= -4 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= -3 \end{aligned}$$

Since there are no 5-tuples that could satisfy these equations, the system is inconsistent.

Suppose we start with $b = (1, -1, 1, 3, 4)^t$ we obtain then the reduction process will yield the echelon-form augmented matrix with the last column $= (1, 3, 0, 0, 0)^t$ and the last two equations of the reduced system will be satisfied for any 5-tuple. Thus the solution set will be the set of all 5-tuples satisfying the first three equations.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \\ x_3 + x_4 + 2x_5 &= 0 \\ x_5 &= 3. \end{aligned}$$

The variables corresponding to the first nonzero elements in each row of the reduced matrix will be referred to as lead variables. Thus x_1, x_3, and x_5 are the lead variables. The remaining variables corresponding to the columns skipped in the reduction process will be referred to as free variables. Hence, x_2 and x_4 are the free variables. If we transfer the free variables over to the right-hand side of this, we obtain the system

$$\begin{aligned} x_1 + x_3 + x_5 &= 1 - x_2 - x_4 \\ x_3 + 2x_5 &= -x_4 \\ x_5 &= 3. \end{aligned}$$

System this is strictly triangular in the unknowns x_1, x_3, and x_5. Thus, for each pair of values assigned to $x_2 = \alpha$ and $x_4 = \beta$, there will be a unique solution.

$$x_5 = 3, \quad x_3 = -\beta - 6, \quad x_1 = -2 - \alpha - 2\beta$$

Let $b = (0, 0, 0, 0, 0)^t$. The reduced echelon form yields

$$\begin{aligned} &x_1 + x_3 + x_5 = -x_2 - x_4 \\ &x_3 + 2x_5 = -x_4 \\ &x_5 = 0 \end{aligned}$$

Thus, we have

$$N(A) = \{(-\alpha - 2\beta, \alpha, -\beta, \beta, 0)^t : \ \alpha, \ \beta \in R\} = \alpha\,(-1, 1, 0, 0, 0) + \beta\,(-2, 0, -1, 1, 0).$$

EXAMPLE (traffic) The augmented matrix for the system

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ -1 & 0 & 0 & 1 & -330 \end{pmatrix}$$

is reduced to

$$\rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & -1 & 0 & 1 & -170 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & -0 & -1 & 1 & -210 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & -0 & 0 & 0 & 0 \end{pmatrix}.$$

The system is **consistent**, and since there is a free variable, there are many possible solutions. The traffic flow diagram does not give enough information to determine x_1, x_2, x_3, and x_4 uniquely. If the amount of traffic were known between any pair of intersections, the traffic on the remaining arteries could easily be calculated. For example, if the amount of traffic between intersections C and D averages 200 automobiles per hour, then $x_4 = 200$. Using this value, we can then solve for x_1, x_2, and x_3 by back substitution $x_1 = 530$, $x_2 = 170$ $x_3 = 410$.

EXAMPLE (Underdetermined) Consider

$$x_1 + x_2 + x_3 + x_4 + x_5 = 2, \quad x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3, \quad x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 2$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

It is **consistent**. We put the free variables $x_2,\ x_3$ over on the right-hand side, it follows that

$$x_1 = 1 - x_2 - x_3,\ x_4 = 2,\ x_5 = -1.$$

Thus, for any real numbers α and β, the 5-tuple

$$(1 - \alpha - \beta, \alpha, \beta, 2, -1)$$

is a solution of the system.

$$N(A) = (-\alpha - \beta, \alpha, \beta, 0, 0) = \alpha\,(-1, 1, , 0, 0, 0) + \beta\,(-1, 0, 1, 0, 0).$$

EXAMPLE (Overdetermined)

$$x_1 + 2x_2 + x_3 = 1,\ 2x_1 - x_2 + x_3 = 2,\ 4x_1 + 3x_2 + 3x_3 = 4,\ 3x_1 + x_2 + 2x_3 = 3$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It is **consistent**. We put the free variable x_3 over on the right-hand side, it follows that

$$x_2 = -\frac{1}{5}x_3,\quad x_1 = 1 - 2x_2 - x_3 = 1 - \frac{1}{5}x_3$$

Thus, for any real number α , the 4-tuple

$$(1 - \frac{1}{3}\alpha, -\frac{1}{3}\alpha, \alpha)$$

is a solution of the system.

$$N(A) = (-\frac{1}{3}\alpha, -\frac{1}{3}\alpha, \alpha) = \alpha\,(-\frac{1}{3}, -\frac{1}{3}, 1).$$

2.5.1 Elementary matrix multiplication and LU decomposition of A

The elementary row operations of Gauss elimination can be rewritten in a matrix product form $A = LU$ where L is a lower triangular matrix and U is a reduced upper triangular matrix. Recall the Gauss-elimination use $(N-1)$ steps to reduce A into U. That is, given an $N \times N$ matrix $A = (a_{i,j})_{1 \le i,j \le N}$, define $A^{(0)} = A$. At n-th step we eliminate the matrix elements below the main diagonal in the n-th column of $A^{(n-1)}$ by adding to the i-th row of this matrix the n-th row multiplied by

$$-\ell_{i,n} := -\frac{a_{i,n}^{(n-1)}}{a_{n,n}^{(n-1)}},\quad i = n+1, \ldots, N.$$

This can be done by multiplying $A^{(n-1)}$ to the left with the lower triangular matrix

$$L_n = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -l_{n+1,n} & & \\ & & \vdots & \ddots & \\ & & -l_{N,n} & & 1 \end{pmatrix},$$

We set

$$A^{(n)} := L_n A^{(n-1)} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -l_{n+1,n} & & \\ & & \vdots & \ddots & \\ & & -l_{N,n} & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ 0 & 0 & a_{n,n} & \cdots & a_{n,N} \\ 0 & 0 & a_{n+1,n} & & a_{n+1,N} \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & a_{N,n} & \cdots & a_{N,N} \end{pmatrix}.$$

which coincides with the n-th Gauss elimination step and n-th step matrix $A^{(n)}$ **has n-th column with all zeros under a_{nn}, i.e., $a^{(n)}_{i,n}$, $n+1 \leq i \leq N$.** After $N-1$ steps, we eliminated all the matrix elements below the main diagonal, so we obtain an upper triangular matrix $U = A^{(N-1)}$. We find the LU decomposition $A = LU$, i.e.,

$$U = A^{(N-1)} = L_{N-1}L_{N-2}\cdots L_1 A, \quad L = (L_{N-1}L_{N-2}\cdots L_1)^{-1} = L_1^{-1}L_2^{-1}\cdots L_{N-1}^{-1}.$$

Because the inverse of a lower triangular matrix L_n is again a lower triangular matrix, and the multiplication of two lower triangular matrices is again a lower triangular matrix, it follows that L is a lower triangular matrix. Moreover, it can be seen that

$$L = \begin{pmatrix} 1 & & & & & \\ l_{2,1} & \ddots & & & & \\ & \ddots & 1 & & & \\ \vdots & \dots & l_{n+1,n} & 1 & & \\ & & \vdots & \ddots & \ddots & \\ l_{N,1} & \dots & l_{N,n} & \dots & l_{N,N-1} & 1 \end{pmatrix}.$$

It is clear that in order for this algorithm to work, one needs to have $a^{(n-1)}_{n,n} \neq 0$ at each step (see the definition of $l_{i,n}$). If this assumption fails at some point, one needs to interchange n-th row with another row below it before continuing (Pivoting). This is why an LU decomposition in general looks like $A = PLU$. (P is a permutation matrix).

Remark **If all diagonal entries of U are nonzero, then $Ax = b$ for $a \in R^{n\times n}$ has a unique solution by back substitution and if $Ax = \vec{0}$, then $x = \vec{0}$, equivalently $N(A) = \{\vec{0}\}$ and column vectors of A are linearly independent**

2.6 Basis and dimension

LEARNING OBJECTIVES FOR THIS SECTION: Basis and dimension of subspace and Gauss elimination. Examples including $N(A)$, $R(A)$ and Properties and Algorithms.

Definition (1) A basis for a subspace S is **a set of linearly independent vectors** whose span is S. The number n of vectors in a basis of the finite-dimensional subspace S is called the dimension of S and we write $dim(S) = n$.
(2) The column **rank** of matrix A is the dimension of the column space of

$$A = [\vec{v}_1|\cdots|\vec{v}_k],$$

where $S = span(\vec{v}_1, \cdots, \vec{v}_k)$. Ref: MATLAB **rank**.
(3) A basis of V is a list of vectors in V that is linearly independent and spans V. The number n of elements in a basis is always equal to the geometric dimension of the subspace S.

Any spanning set for a subspace can be changed into a basis by removing redundant vectors (column wise Gauss elimination (see algorithms, below).

$$S = span(\vec{v}_1, \cdots \vec{v}_n) \text{ and } \dim(S) = n$$

and there exists a unique $a_1, \cdots a_n \in R$ such that every $s \in S$ is represented by

$$s = a_1\,\vec{v}_1 + \cdots a_n\vec{v}_n.$$

The dimension of the null space $N(A)$ is called the nullity of the matrix, and is related to the rank of the matrix A by the following equation:

$$\text{rank}(A) + \text{nullity}(A) = m,$$

which is known as the rank-nullity theorem. In fact for $rank(A) = n \le m$.

$$\vec{v}_{n+i} = a_{i1}\vec{v}_1 + \cdots + a_{in}\vec{v}_n, \quad 1 \le i \le k - n.$$

Thus, $\dim(N(A)) = m - n$.
EXAMPLE Let S be the subspace of R^4 defined by the equations

$$x_1 = 2x_2 \text{ and } x_3 = 5x_4.$$

Then the vectors $(2, 1, 0, 0)$ and $(0, 0, 5, 1)$ are a basis for S. In particular, every vector that satisfies the above equations can be written uniquely as a linear combination of the two basis vectors:

$$(2t_1, t_1, 5t_2, t_2) = t_1(2, 1, 0, 0) + t_2(0, 0, 5, 1).$$

The subspace S is two-dimensional. Geometrically, it is the plane in R^4 passing through the points $(0, 0, 0, 0)$, $(2, 1, 0, 0)$, and $(0, 0, 5, 1)$.

EXAMPLE

In many applications, it is necessary to find a particular subspace of a vector space $V = R^4$. This can be done by finding a set of basis elements of the subspace. For example, to find all solutions of the system

$$x_1 + x_2 + x_3 = 0, \;\; 2x_1 + x_2 + x_4 = 0$$

we must find the null space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{pmatrix}.$$

and we have

$$x_1 + x_2 + x_3 = 0, \;\; -x_2 - 2x_3 + x_4 = 0$$

We choose x_3 and x_4 as free variables and solve for $x_1, \; x_2$,

$$x_2 = -2x_3 + x_4, \;\; x_1 = -x_2 - x_3 = -x_3 - x_4.$$

Thus, we obtain a basis of $N(A)$

$$\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

which corresponds to $x_3 = 1, \; x_4 = 0$ and $x_3 = 0, \; x_4 = 1$, respectively.

In general we have

Basis for a null space $N(A)$ Recall $N(A) = \{x \in R^n : Ax = \vec{0}\}$ is a subspace of matrix $A \in R^{m\times n}$. One can use Gauss elimination to find a basis of $N(A)$.

- Use elementary row operations to put A in reduced row echelon form.
- Using the reduced row echelon form, determine which of the variables $x_1, x_2, \cdots, x_k$ are free. Write equations for the dependent variables in terms of the free variables.
- For each free variable x_i, choose a vector in the null space for which $x_i = 1$ and the remaining free variables are zero. The resulting collection of vectors is a basis for the null space of A.

EXAMPLE The standard basis for R^3 is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$; however, there are many bases that we could choose for R^3.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\},$$

Standard Bases We refer to the set $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ as the standard basis for R^3. We refer to this basis as the standard basis because it is the most natural one to use for representing vectors in R^3. More generally, the standard basis for R^n is the set $\{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\}$ since

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\,\vec{e}_1 + \cdots + x_n\,\vec{e}_n$$

The most natural way to represent matrices in $R^{2\times 2}$ is in terms of the standard 2×2 basis matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The standard way to represent a polynomial in P_n is in terms of the standard basis functions $\{1, x, x^2, ..., x^n\}$, i.e.,

$$p(x) = a_0 + a_1 x + \cdot a_n x^n$$

In general, we have

Theorem 1 If $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a spanning set for a vector space V, then any collection of m vectors in V, where $m > n$, are linearly dependent.

Proof: Let $\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_m\}$ be m vectors in V where $m > n$. Then, since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ span V, we have

$$\vec{u}_i = a_{1i}\vec{v}_1 + a_{2,i}\vec{v}_2 + \cdots a_{n,i}\vec{v}_n$$

Thus,

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots c_m\vec{u}_m = \sum_{i=1}^{m} c_i(\sum_{j=1}^{n} a_{ji}\vec{v}_j) = \sum_{j=1}^{n}(\sum_{i=1}^{m} a_{ji}c_i)\vec{v}_j$$

Now consider the system of equations

$$\sum_{i=1}^{m} a_{ji}c_i = A\vec{c} = \vec{0}.$$

This is a homogeneous system with more unknowns than equations. Therefore, the system must have a nontrivial solution $(c_1, c_2, \cdots, c_m)^t$. Thus, $\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_m\}$ are linearly dependent. □

Theorem 2 If V is a vector space of dimension $n > 0$, then

(I) any set of n linearly independent vectors spans V .

(II) any n vectors that span V are linearly independent.

Proof: Suppose that $\{\vec{v}_1, \cdots, \vec{v}_n\}$ are linearly independent and $\vec{v}$ is any other vector in V. Since V has dimension n, it has a basis consisting of n vectors and these vectors span V. It follows from Theorem 1 that $\{\vec{v}_1, \cdots, \vec{v}_n\, \vec{v}\}$ must be linearly dependent. Thus there exists $c_i \in R,\ 1 \le i \le n+1$ not all zero such that

$$c_1\vec{v}_1 + \cdots c_n\vec{v}_n + c_{n+1}\vec{v} = \vec{0}$$

Then, c_{n+1} cannot be zero since if $c_{n+1} = 0$, the $c_i = 0,\ 1 \le i \le n$.

$$\vec{v} = -\frac{c_1}{c_{n+1}}\vec{v}_1 + \cdots - \frac{c_n}{c_{n+1}}\vec{v}_n$$

To prove (II), suppose that $span(\vec{v}_1, \cdots, \vec{v}_n) = V$. If $\{\vec{v}_1, \cdots, \vec{v}_n\}$ are linearly dependent, then one of the $\vec{v}_i$s, say $\vec{v}_n$, can be written as a linear combination of the others. i.e., $\dim(V) < n$, which is a contradiction. □

Theorem 3 The dimension of the sum satisfies the inequality

$$\max(\dim W_1, \dim W_2) \leq \dim(W_1 + W_2) \leq \dim(W_1) + \dim(W_2).$$

Here the minimum only occurs if one subspace is contained in the other, while the maximum is the most general case. The dimension of the intersection and the sum are related:

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Proof: Let $\{\vec{u}_1; \vec{u}_2, \cdots \vec{u}_m\}$ be a basis of $W_1 \cap W_2$, thus $\dim(W_1 \cap W_2) = m$. Because $\{\vec{u}_1; \vec{u}_2, \cdots \vec{u}_m\}$ is a basis of $W_1 \cap W_2$, it is linearly independent in W_1. Hence this list can be extended to a basis $\{\vec{u}_1; \vec{u}_2, \cdots \vec{u}_m, \vec{v}_1, \cdots, \vec{v}_j\}$ of W_1. where $\dim(W_1) = m + j$. Also extend $\{\vec{u}_1; \vec{u}_2, \cdots \vec{u}_m\}$ to a basis $\{\vec{u}_1; \vec{u}_2, \cdots \vec{u}_m, \vec{w}_1, \cdots, \vec{w}_k\}$ of W_2 and thus $\dim(W_2) = m + k$. We will show that $\{\vec{u}_1; \vec{u}_2, \cdots, \vec{u}_m, \vec{v}_1, \cdots, \vec{v}_j, \vec{w}_1, \cdots, \vec{w}_k\}$ is a basis of $W_1 + W_2$. This will complete the proof, because then we will have

$$\dim(W_1 + W_2) = m + j + k = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = m + j + m + k - m$$

Clearly $span(\{\vec{u}_1; \vec{u}_2, \cdots \vec{u}_m, \vec{v}_1, \cdots, \vec{v}_j, \vec{w}_1, \cdots, \vec{w}_k\})$ is contained in W_1 and W_2 and and hence equals $W_1 + W_2$. Suppose

$$a_1\vec{u}_1 + \cdots + a_m\vec{u}_m + b_1\vec{v}_1 + \cdots + b_j\vec{v}_l + c_1\vec{w}_1 + \cdots + c_k\vec{w}_k = \vec{0}, \tag{2.1}$$

Then, it rewritten as

$$c_1\vec{w}_1 + \cdots + c_k\vec{w}_k = -(a_1\vec{u}_1 + \cdots + a_m\vec{u}_m + b_1\vec{v}_1 + \cdots + b_j\vec{v}_j),$$

which shows that $c_1\vec{w}_1 + \cdots + c_k\vec{w}_k \in W_1 \cap W_2$. Since $\{\vec{u}_1; \vec{u}_2, \cdots \vec{u}_m\}$ is a basis of $W_1 \cap W_2$,

$$c_1\vec{w}_1 + \cdots + c_k\vec{w}_k = d_1\vec{u}_1 + \cdots + d_m\vec{u}_m$$

for some $d_1, \cdots, d_m$. But since $\{\vec{u}_1; \vec{u}_2, \cdots \vec{u}_m, \vec{w}_1, \cdots, \vec{w}_k\}$ are linearly independent, all coefficients c's and d's are zero. Thus, our original equation becomes

$$a_1\vec{u}_1 + \cdots + a_m\vec{u}_m + b_1\vec{v}_1 + \cdots + b_j\vec{v}_j = \vec{0}$$

Since $\{\vec{u}_1; \vec{u}_2, \cdots \vec{u}_m, \vec{v}_1, \cdots, \vec{v}_j\}$ are linearly independent all a's and b's equal to zero. Thus, $\{\vec{u}_1; \vec{u}_2, \cdots \vec{u}_m, \vec{v}_1, \cdots, \vec{v}_j, \vec{w}_1, \cdots, \vec{w}_k\}$ is a basis of $W_1 + W_2$, which completes the proof. □

Remark If $W_1 \cap W_2 = \{\vec{0}\}$, then $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$. Notation: In this case $W_1 \oplus W_2$.

Find a basis for $W = span(\vec{a}_1, \cdots \vec{a}_n)$. Let

$$A = \begin{pmatrix} \vec{a}_1^t \\ \vdots \\ \vec{a}_n^t \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ \vdots & \vdots & & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{pmatrix}$$

Using elementary row operations, this matrix is transformed to the row echelon form. Then, it has the following shape:

$$\begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,m} \\ \vdots & \vdots & & \vdots \\ c_{q,1} & c_{q,2} & \cdots & c_{q,m} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Then,

$$\{\vec{c}_1, \cdots \vec{c}_q\} \text{ is a basis of } W \text{ and } \dim(W) = q.$$

Zassenhaus algorithm Algorithm for finding bases for intersection $W_1 \cap W_2$ and sum $W_1 + W_2$. Assume

$$W_1 = span(\vec{a}_1, \cdots \vec{a}_n), \;\; W_2 = span(\vec{b}_1, \cdots \vec{b}_k)$$

subspaces of R^m and let

$$A = \begin{pmatrix} \vec{a}_1^t \\ \vdots \\ \vec{a}_n^t \end{pmatrix} \;\; B = \begin{pmatrix} \vec{b}_1^t \\ \vdots \\ \vec{b}_k^t \end{pmatrix}.$$

The algorithm creates the following block matrix of size $((n+k) \times (2m)) \times ((n+k) \times (2m))$:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} & a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} & a_{n,1} & a_{n,2} & \cdots & a_{n,m} \\ b_{1,1} & b_{1,2} & \cdots & b_{1,m} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_{k,1} & b_{k,2} & \cdots & b_{k,m} & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} A & A \\ & \\ B & 0 \end{pmatrix}.$$

Using elementary row operations, this matrix is transformed to the row echelon form. Then, it has the following shape:

$$\begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,m} & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{q,1} & c_{q,2} & \cdots & c_{q,m} & * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & d_{1,1} & d_{1,2} & \cdots & d_{1,m} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & d_{\ell,1} & d_{\ell,2} & \cdots & d_{\ell,m} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Here, $*$ stands for arbitrary numbers, and the vectors . Then

$$\{\vec{c}_1, \cdots \vec{c}_q\} \text{ is a is a basis of } W_1 + W_2.$$

and

$$\{\vec{d}_1, \cdots \vec{d}_\ell\} \text{ is a is a basis of } W_1 \cap W_2.$$

EXAMPLE

$$W_1 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

and

$$W_2 = \left\{ \begin{pmatrix} 5 \\ 0 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ -3 \\ -2 \end{pmatrix} \right\}$$

of the vector space R^4. Using the standard basis, we create the following matrix of dimension $(2+2) \times (2 \cdot 4)$:

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ \\ 5 & 0 & -3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using elementary row operations, we transform this matrix into the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & * & * & * & * \\ 0 & 1 & 0 & -1 & * & * & * & * \\ 0 & 0 & 1 & -1 & * & * & * & * \\ \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

(some entries have been replaced by $*$ because they are irrelevant to the result). Therefore,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

is a basis of $W_1 + W_2$ and

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis of $W_1 \cap W_2$.

MATLAB implementation

Given matrix $A \in R^{m \times n_1}$ and $A \in R^{m \times n_2}$ use matlab LU decomposition:

```
[L,U,P]=lu([[A';B'] [A';0*B']]); U
```

where U is a resulting upper triangular form we are looking for. Try it with

```
A=rand(4,2), B=[sum(A,2) rand(4,1)];
```

2.7 Inverse of matrix A

LEARNING OBJECTIVES FOR THIS SECTION: Linear equation, Inverse of matrix, Gauss elimination. Nonsingular and Singular matrixes.

Let $I = I_n \in R^{n\times n}$ be the identity matrix = diagonal matrix with diagonal entries are all one. Then, $I_nA = AI_n = A$ for all $A \in R^{n\times n}$.

Definition (**Inverse of matrix A**) Let $A \in R^{n\times n}$ be a square matrix. A matrix $B \in A \in R^{n\times n}$ is an inverse of A if

$$AB = I_n \text{ identity matrix .}$$

and denoted by $B = A^{-1}$., i.e.

$$AA^{-1} = I_n \tag{2.2}$$

If so, A is non singular.

Recall that

$$x = A^{-1}b \text{ satisfies a linear equation } Ax = b \text{ for all } b \in R^n.$$

In fact,

$$Ax = A(A^{-1}b) = (AA^{-1})b = I_nb = b.$$

Note that if $\tilde{B} \in R^{n\times n}$ satisfies $\tilde{B}A = I$, then

$$\tilde{B} = \tilde{B}I = \tilde{B}(AB) = (\tilde{B}A)B = IB = B$$

and thus

$$A^{-1}A = I_n. \tag{2.3}$$

Theorem **Inverse of product AB**

If $A,\ B \in R^{n\times n}$ are nonsingular , then

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n.$$

Definition (**Transpose)**

The transpose of an $m \times n$ matrix A, denoted by A^t, is the $n \times m$ matrix such that the (j, i)-entry is given by $A_{i,j}$

$$(A^t)_{j,i} = A_{i,j} \text{ for } i = 1, \cdots, m \text{ and } j = 1, \cdots, n.$$

In other words, column i of A^t comes from row i of A, or equivalently row j of A^t comes from column j of A.

The following properties hold

$$(A^t)^t = A,\ \ (A + B)^t = A^t + B^t,\ \ (AB)^t = B^tA^t \text{ and } (A^t)^{-1} = (A^{-1})^t$$

In fact, since

$$(AB)_{i,j} = \sum_k a_{ik} b_{k,j}, \quad (B^t A^t)_{i,j} = \sum_k b_{k,i} a_{k,j},$$

$$((AB)^t)_{i,j} = \sum_k a_{jk} b_{k,i} = (B^t A^t)_{i,j}$$

For $B = A^{-1}$

$$I_n = I_n^t = (AB)^t = B^t A^t \Rightarrow (A^t)^{-1} = B^t = (A^{-1})^t \text{ by (2.2)–(2.3).}$$

How to find A^{-1} by Gauss Elimination Form an attached matrix

$$[A \mid I_n]$$

Then apply a Gauss-Jordan deduction and we obtain the reduced matrix (row echelon form)

$$[U \mid C],$$

where U is the reduced upper matrix of A. Then

$$A^{-1} = U^{-1} C.$$

EXAMPLE Find all values of a such that the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & a \\ 0 & a & -1 \end{pmatrix}$$

is invertible. Solution: Gauss elimination of A:

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & a \\ 0 & a & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & a+2 \\ 0 & a & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & a+2 \\ 0 & 0 & -a^2-2a-1 \end{pmatrix}$$

implies that

$$-a^2 - 2a - 1 = -(a+1)^2 \neq 0 \to a \neq -1$$

EXAMPLE A block matrix formula:

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1} B C^{-1} \\ O & C^{-1} \end{pmatrix}$$

where $A \in R^{n \times n}$, $B \in R^{n \times m}$ and $C \in R^{m \times m}$. Solution:

$$\begin{pmatrix} A^{-1} & -A^{-1} B C^{-1} \\ O & C^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ O & C \end{pmatrix} = \begin{pmatrix} A^{-1} A & A^{-1} B - A^{-1} B C^{-1} C \\ O & C^{-1} C \end{pmatrix}^{-1} = \begin{pmatrix} I_n & O \\ O & I_m \end{pmatrix}$$

Equivalently, it is equivalent find a solution to

$$Ax + By = a \ \ Cy = b.$$

i.e., $y = C^{-1} b, \ \ x = A^{-1}(a - BC^{-1} b)$ by back substitution. Equivalently,

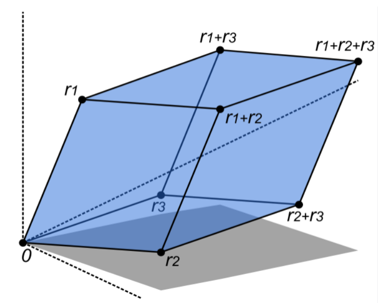
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^{-1} & -A^{-1} B C^{-1} \\ O & C^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Remark **A is non singular iff the reduced triangle matrix U has nonzero diagonals, i.e., $U^{-1} C$ is carried out by backward substitution for each column vector of C.**

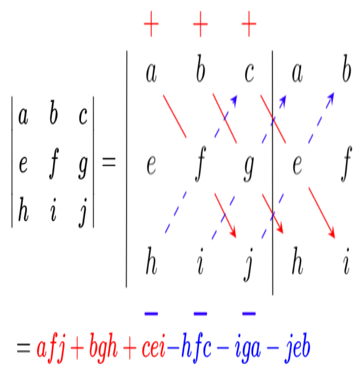
3 Determinant and Matrix inverse

LEARNING OBJECTIVES FOR THIS Chapter: Determinant, Cramers rule for inverse of matrix A. Cofactors and minors of A. Inverse matrix, Properties of Determinant. Alternative to Gasss-Jordan reduction to upper triangular matrix

In linear algebra, the determinant is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the linear transformation described by the matrix. The determinant of a matrix A is denoted $det(A)$ or $|A|$. Geometrically, it can be viewed as the volume scaling factor of the linear transformation described by the matrix. This is also **the signed volume of the n-dimensional parallelepiped spanned by the column or row vectors of the matrix**. The determinant is positive or negative according to whether the linear transformation preserves or reverses the orientation of a real vector space.



The volume of this parallelepiped is the absolute value of the determinant of the matrix formed by the columns constructed from the vectors r1, r2, and r3.



In the case of a 2×2 matrix the determinant may be defined as

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Similarly, for a 3×3 matrix A, its determinant is

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - bdi - afh.$$

Each determinant of a 2×2 matrix in this equation is called a minor of the matrix A. This procedure can be extended to give a recursive definition for the determinant of an $n \times n$ matrix, the Laplace expansion.

The following scheme (rule of Sarrus) for calculating the determinant of a 3×3 matrix, the sum of the products of three diagonal north-west to south-east lines of matrix elements, minus the sum of the products of three diagonal south-west to north-east lines of elements, when the copies of the first two columns of the matrix are written beside it as in the illustration:

Definition (Determinant) The determinant of an $n \times n$ matrix A, denoted $\det(A)$, is a scalar associated with the matrix A that is defined inductively as

$$\det(A) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n > 1. \end{cases}$$

where

$$A_{i,j} = (-1)^{1+j} \det(M_{ij}).$$

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}.$$

Laplace expansion expresses the determinant of a matrix in terms of its minors. The cofactor A_{ij} is defined to be the determinant of the $(n-1)\times(n-1)$-matrix **minor** M_{ij} that results from A by removing the i-th row and the j-th column. The expression $(-1)^{i+j}\det(M_{ij})$ is known as a **cofactor**.

Equivalent Definition (Leibniz formula) The determinant of a $n \times n$ matrix A is the scalar quantity

$$\det(A) = \sum_{\phi \in S_n} \operatorname{sign}(\phi) a_{1\phi(1)} a_{2\phi(2)} \cdots a_{n\phi(n)}$$

where S_n is all permutations of indices $(1, 2, \cdots, n)$ and $\operatorname{sign}(\phi)$ is the sign of permutation (reordering) ϕ. If ϕ requires s interchanges of indices $(1, 2, \cdots, n)$, then $\operatorname{sign}(\phi) = (-1)^s$.

In fact, we have

$$\det(A) = \textstyle\sum_{k=1}^n \sum_{\phi(k)=1,\ \phi\in S_n} \operatorname{sign}(\phi) a_{1\phi(1)} a_{2\phi(2)} \cdots a_{n\phi(n)}$$

$$= \textstyle\sum_{k=1}^n a_{k1} \sum_{\phi(1:k-1:k+1:n)\in S_{n-1}} \operatorname{sign}(\phi) a_{1\phi(1)} \cdots a_{1\phi(k-1)} a_{1\phi(k+1)} \cdots a_{n,\phi(n)}$$

$$= a_{11}A_{11} + a_{21}A_{21} + \cdots + a_{n1}A_{n1}$$

For example, $\{S_2 = (1, 2),\ (2, 1)\}$.

$$S_3 = \{(1, 2, 3),\ (1, 3, 2),\ (2, 1, 3),\ (2, 3, 1),\ (3, 1, 2),\ (3, 2, 1)\}$$

S_n contains $n!$ elements..

Remark (**column-wise**)

$$\det(A) = a_{11}A_{11} + a_{21}A_{21} + \cdots + a_{n1}A_{n1}.$$

Remark (Expansion at k the row (column) By interchanging the first row (column) with the kth row (column of A), we have

$$\det(A) = -(a_{k1}A_{k1} + a_{k2}A_{k2} + \cdots + a_{kn}A_{kn})$$

$$\det(A) = -(a_{1k}A_{1k} + a_{2k}A_{2k} + \cdots + a_{nk}A_{nk}).$$

A number of properties relate to the effects on the determinant of changing particular rows or columns, which all follow from the Laplace expansion and Leibniz formula.

(1) Viewing an $n \times n$ being composed of n columns, the determinant is an n-linear function. This means that if the j-th column of a matrix A s written as a sum $a_j = v + w$ of two column vectors v, w , and all other columns are left unchanged, then the determinant of A is the sum of the determinants of the matrices obtained from A by replacing the j-th column by v, denoted by A_v then by w, denoted by denoted A_w (and a similar relation holds when writing a column as a scalar multiple of a column vector).

$$\begin{aligned}\det(A) &= \det([\mathbf{a}_1|\ldots|\mathbf{a}_j|\ldots|\mathbf{a}_n]) \\ &= \det([\ldots|\mathbf{v}+\mathbf{w}|\ldots]) \\ &= \det([\ldots|\mathbf{v}|\ldots]) + \det([\ldots|\mathbf{w}|\ldots]) \\ &= \det(A_v) + \det(A_w)\end{aligned}$$

(2) If in a matrix, any row or column has all elements equal to zero, then the determinant of that matrix is 0. This n-linear function is an alternating form. This means that whenever two columns of a matrix are identical, or more generally some column can be expressed as a linear combination of the other columns (i.e. the columns of the matrix form a linearly dependent set), its determinant is 0.

Above all properties for columns have their counterparts in terms of rows: viewing an $n \times n$ matrix as being composed of n rows, the determinant is an n-linear function.

(3) **Whenever two rows of a matrix are identical, its determinant is 0**.

(4) **Interchanging any pair of columns or rows of a matrix multiplies its determinant by -1**. This follows from more generally, any permutation of the rows or columns multiplies the determinant by the sign of the permutation. By permutation, it is meant viewing each row as a vector R_i (equivalently each column as C_i) and reordering the rows (or columns) by interchange of R_j and R_k (or C_j and C_k), where j, k are two indices chosen from 1 to n for an $n \times n$ square matrix.

(5) Adding a scalar multiple of one column to another column does not change the value of the determinant. since the determinant changes by a multiple of the determinant of a matrix with two equal columns, which determinant is 0. Similarly, **adding a scalar multiple of one row to another row leaves the determinant unchanged**.

For example, the determinant of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix}$$

can be computed using the following matrices (Gauss eliminations):

$$B = \begin{bmatrix} -2 & 2 & -3 \\ 0 & 0 & 4.5 \\ 2 & 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 2 & -3 \\ 0 & 0 & 4.5 \\ 0 & 2 & -4 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 2 & -3 \\ 0 & 2 & -4 \\ 0 & 0 & 4.5 \end{bmatrix}.$$

Here, B is obtained from A by adding $-1/2\times$ the first row to the second, so that $\det(A) = \det(B)$. C is obtained from B by adding the first to the third row, so that $\det(C) = \det(B)$. Finally, D is obtained from C by exchanging the second and third row, so that $\det(D) = -\det(C)$. The determinant of the (upper) triangular matrix D is the product of its entries on the main diagonal: $(-2) \cdot 2 \cdot 4.5 = -18$. Therefore, $\det(A) = -\det(D) = +18$.

<u>Remark</u> If row vectors of A are linearly dependent, $\det(A) = 0$, Conversely, if row vectors of A are linearly independent if and if $\det(A) \neq 0$.

<u>**Definition**</u> A matrix $A \in R^{n\times n}$ is singular if $\det(A) = 0$, otherwise is **non-singular**.

<u>Theorem 1</u> If A is an $n \times n$ matrix, then $\det(A^t) = \det(A)$.

<u>Theorem 2</u> $\det(cA) = c^n \det(A)$.

<u>Theorem 3</u> For all elementary operations E, $\det(EA) = \det(E)\det(A)$ and

$$\det(A) = \det(L)\det(U) \text{ where } L = E_1E_2\cdots E_{n-1}$$

Proof: For **elementary row operations [I], [II] and [III]**

$$\det(EA) = \det(E)\det(A).$$

<u>Theorem 4</u> If U is an upper triangular matrix $\det(U) = u_{11} \times \cdots \times u_{nn}$,

<u>Theorem 5</u> $\det(AB) = \det(A)\det(B)$.
Proof: Assume B is nonsingular and

$$B = LU = E_1E_2\cdots E_{n-1}diag(U)\tilde{E}_1\cdots\tilde{E}_{n-1},$$

where E_k are elementary row operation and $\tilde{E}_j$ are elementary column operations. That is,

$$U^t = \tilde{E}_1^t\cdots\tilde{E}_{n-1}^t diag(U),$$

where $\tilde{E}_1^t, \cdots, \tilde{E}_{n-1}^t$ is lower triangular matrices of elementary operations. Thus, we have

$$\det(AB) = \det(A)\det(E)\det(diag(U))\det(\tilde{E}) = \det(A)\det(U) = \det(A)\det(B).$$

<u>Corollary</u> $\det(A^{-1}) = \det(A)^{-1}$ since $A^{-1}A = I$.

EXAMPLE Expanding by a row or column can sometimes be a quick method of evaluating the determinant of matrices containing a lot of zeros. For example, let

$$A = \begin{bmatrix} 9 & 0 & 2 & 6 \\ 1 & 2 & 9 & -3 \\ 0 & 0 & -2 & 0 \\ -1 & 0 & -5 & 2 \end{bmatrix}$$

Then, expanding by the third row, we get $\det(A) = -2 \times \det \begin{bmatrix} 9 & 0 & 6 \\ 1 & 2 & -3 \\ -1 & 0 & 2 \end{bmatrix}$ and by the second column, $\det(A) = -2 \times 2 \times \det \begin{bmatrix} 9 & 6 \\ -1 & 2 \end{bmatrix} = -96$.

3.1 Cramer's rule

For a matrix equation $Ax = b$, given that A has a nonzero determinant, the solution $x = A^{-1}b$ is given by Cramer's rule:

$$x_i = \frac{\det(A_i)}{\det(A)}, \; i = 1, 2, 3, \cdots, n$$

where A_i is the matrix formed by replacing the i-th column of A by the column vector b. This follows immediately by column expansion of the determinant, i.e.

$$\det(A_i) = \det \begin{bmatrix} a_1, & \ldots, & b, & \ldots, & a_n \end{bmatrix} = \sum_{j=1}^{n} x_j \det \begin{bmatrix} a_1, & \ldots, a_{i-1}, & a_j, & a_{i+1}, & \ldots, & a_n \end{bmatrix} = x_i \det(A)$$

where a_j is the j-th column of A since

$$b = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n.$$

The rule is also equivalently written as

$$A \operatorname{adj}(A) = \operatorname{adj}(A) \, A = \det(A) \, I_n.$$

or equivalently

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

where the **adjugate matrix** $adj(A)$ is the transpose of the matrix of the cofactors, that is,

$$(\operatorname{adj}(A))_{ij} = (-1)^{i+j} A_{ji}.$$

In fact i-th column $\vec{x}_i$ of A^{-1} equals to $A^{-1}\vec{e}_i$ where $\vec{e}_i$ is the i the unit vector. By Cramer's rule, the j the coordinate of $\vec{x}_i$ is given by

$$(A^{-1})_{ji} = \frac{\det(A_j)}{\det(A)} = \frac{1}{\det(A)} (-1)^{i+j} A_{ij}.$$

Note that the adjugate matrix $adj(A)$ is the transpose of the cofactors of A.
<u>The rule for 3×3 case:</u>

$$\begin{cases} a_1 x + b_1 y + c_1 z & = d_1 \\ a_2 x + b_2 y + c_2 z & = d_2 \\ a_3 x + b_3 y + c_3 z & = d_3 \end{cases}$$

which in matrix format is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

Then the values of x, y and z can be found as follows:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \text{ and } z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

<u>EXAMPLE</u>

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose $\det(A) = ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

<u>EXAMPLE</u>

$$\det \begin{pmatrix} A & B \\ O & C \end{pmatrix} = \det(A)\det(C)$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$ and $C \in R^{m\times m}$. Solution: Apply the Gauss elimination to A and C to obtain a upper triangular matrix

$$\begin{pmatrix} U_1 & \tilde{B} \\ O & U_2 \end{pmatrix}$$

whose the determinant$= \det(U_1)\det(U_2) = det(A)\det(C)$.

It has recently been shown that Cramer's rule can be implemented in $O(n^3)$ time, which is comparable to more common methods of solving systems of linear equations, such as LU, QR, or singular value decomposition.

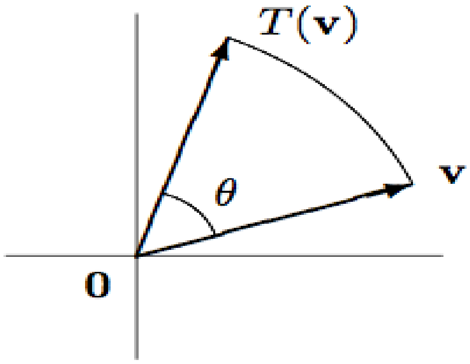
<u>Theorem</u> A matrix A is nonsingular ($\det(A) \neq 0$) if and only if A^{-1} exists. If so, $rank(A) = n$, $Ax = b$ has a unique solution $x = A^{-1}b$ and $N(A) = \{\vec{0}\}$.

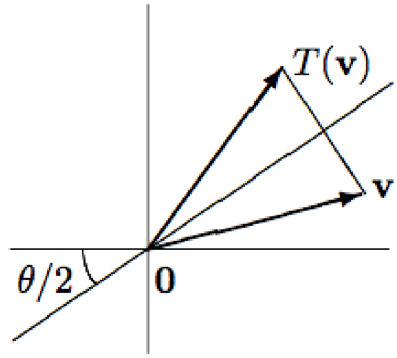
4 Linear Transform

LEARNING OBJECTIVES FOR THIS CHAPTER Fundamental Theorem of Linear Maps Matrix representation and Change of basis and Similarity transform, Inverse map, Injective and Surjective map.

Let V and W be vector spaces with scalars coming from the same field F. A mapping $T : V \to W$ is a linear transformation if for any two vectors x_1 and x_2 in V and any scalar $a_1, a_2 \in F$, the following are satisfied:

$$T(a_1x_1 + a_2x_2) = a_1T(x_1) + a_2T(x_2)$$





Definition (Composition of linear transformations) Let $T_1 \in \mathcal{L}(V, W)$ and $T_2 \in \mathcal{L}(W, U)$. We define a transformation $T_2 T_1 : V \to U$ by $(T_2 T_1)(u) = T_2(T_1(u))$ for $u \in V$ In particular, we define $T^2 = TT$ and $T^{i+1} = T^i T$ for $i > 2$.

EXAMPLE (Matrix) $V = R^n$ and $W = R^m$ and $T(x) = Ax$ for $A \in R^{m \times n}$.

EXAMPLE (Derivative) $T_1 = \frac{d}{dx} = D$ derivative and $V = C^1(a, b)$ and $W = C(a, b)$

$$\frac{d}{dx}(a_1 f_1 + a_2 f_2) = a_1 \frac{d}{dx} f_1 + a_2 \frac{d}{dx} f_2.$$

EXAMPLE (Integration) $T_2 f = \int_0^x f\,dx$ integral and $V = C(a, b)$ and $W = C^1(a, b)$:

$$\int_0^x (a_1 f_1 + a_2 f_2)\,dx = a_1 \int_0^x f_1\,dx + a_2 \int_0^x f_2\,dx..$$

Since $\frac{d}{dx}(\int_0^x f\,dx) = f(x)$, we have

$$T_1 T_2 f = T_1(T_2 f) = f \text{ for } f \in C(a, b)$$

EXAMPLE (Multiplication) $(Tf)(x) = (a+bx+cx^2)f(x)$ for $a,\ b,\ c \in R$. $V = W = C(a, b)$.

EXAMPLE (Composite of Derivative and Multiplication) $(Tf)(x) = x \frac{d}{dx} f$. $V = C^1(a, b)$, $W = C(a, b)$.

EXAMPLE (Shift) Let $V = C(R)$, the space of continuous functions. Every $\alpha \in R$ gives rise to two linear maps, shift $S_\alpha : V \to V$, $S_\alpha(f) = f(x - \alpha)$ and evaluation $E_\alpha V \to R$, $E_\alpha(f) = f(\alpha)$.

Isomorphism identifying V with $\dim(V) = n$ with R^n. Assume $\dim(V) = n$ and $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a linearly independent basis, i.e., every vector $\vec{v} \in V$ is uniquely represented by

$$\vec{v} = a_1\,\vec{v}_1 + \cdots + a_n\,\vec{v}_n$$

That is. $\vec{v} \in V$ corresponds to exactly one such column vector $(a_1, \cdots, a_n)^t$ in R^n, and vice versa. That is, for all intents and purposes, we have just identified the vector space V with the more familiar space R^n.

EXAMPLE $\{1, x, x^2\}$ is the standard basis of P_2:

$$V = P_2 : a + bx + cx^2 \to \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in R^3$$

defines an isomorphism identifying P_2 with R^3.

Matrix representation A of T Assume $\dim(V) = n$ and $\dim(W) = m$. We will now see that we can express linear transformations as matrices as well. Hence, one can simply focus on studying linear transformations of the form $T(x) = Ax$ where $A \in R^{m\times n}$ is a matrix.

In fact, let $\{\vec{v}_1, \cdots, \vec{v}_n\}$ be a basis of V and $\{\vec{w}_1, \cdots, \vec{w}_m\}$ be a basis of W. Then, we have

$$\mathbf{T(\tilde{v}_j) = a_{1j}\,\tilde{w}_1 + \cdots a_{mj}\,\tilde{w}_m}$$

and define $A = (a_{ij}) \in R^{m\times n}$, i.e.,

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

In fact, if $y = Ax \in R^m$, given $x \in R^n$ we have

$$T(x_1\vec{v}_1 + \cdots + x_n\vec{v}_n) = x_1\,T(\vec{v}_1) + \cdots + x_n\,T(\vec{v}_n) = y_1\,\vec{w}_1 + \cdots + y_m\,\vec{w}_m$$

since

$$x_1\,T(\vec{v}_1) + \cdots + x_n\,T(\vec{v}_n) = (a_{11}x_1 + \cdots + a_{1n}x_n)\vec{w}_1 + \cdots + (a_{m1}x_1 + \cdots + a_{mn}x_n)\vec{w}_m.$$

Corollary If $T\mathcal{L}(R^n, R^m)$ then $T(x) = Ax$ if the jth column vector $\vec{a}_j$ of A is given by $\vec{a}_j = T(\vec{e}_j),\ j = 1, \cdots, n$.

EXAMPLE Consider the linear transformation $D : P_2 \to P_1$ that sends f to $\frac{d}{dx}$. Then, the matrix representation A of D, $V = P_2$ and $W = P_1$ with the standard basis $\{1, x, x^2\}$ is given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in R^{2\times 3}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ 2c \end{pmatrix}.$$

This represents the fact that $\frac{d}{dx}(a + bx + cx^2) = b + 2cx$.

EXAMPLE Consider the integral map $T_2 : P_2 \to P_3$ that sends f to $\int_0^x f\,dx$. Then, the matrix representation A of T_2, $V = P_2$ and $W = P_3$ with the standard basis $\{1, x, x^2, x^3\}$ is given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \in R^{4\times 3}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ \frac{1}{2}b \\ \frac{1}{3}c \end{pmatrix}$$

This represents the fact that $\int_0^x (a + bx + cx^2) = ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3$.

EXAMPLE Consider the integral map $T : P_3 \to P_3$ that sends f to $x\frac{d}{dx}f$. Then, the matrix representation A of T_2, $V = P_3$ and $W = P_3$ with the standard basis $\{1, x, x^2, x^3\}$ is given by

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \in R^{4\times 4}, \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 2c \\ 3d \end{pmatrix}$$

This represents the fact that $x\frac{d}{dx}(a + bx + cx^2 + dx^3) = bx + 2cx^2 + 3dx^3$.

EXAMPLE $T : R^2 \to R^2$, $T = R_\theta$ is a rotation by θ anti-clockwise about the origin. Since $T(1, 0) = (\cos\theta, \sin\theta)$ and $T(0, 1) = (-\sin\theta, \cos\theta)$,

$$T\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha\cos\theta - \beta\sin\theta \\ \alpha\sin\theta + \beta\cos\theta \end{pmatrix},$$

so the matrix using the standard bases is

$$A = \begin{pmatrix} cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Now clearly R_θ followed by R_ϕ is equal to $R_{\theta+\phi}$. Thus

$$R_\phi R_\theta = \begin{pmatrix} cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -\cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\ cos\theta + \cos\phi\sin\theta & -sin\phi\sin\theta + \cos\phi\cos\theta \end{pmatrix}$$

$$= R_{\phi+\theta} = \begin{pmatrix} cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix}$$

which derives the addition formulae for sin and cos.

EXAMPLE $T : P^2 \to R^2$, T is evaluation $Tf = (f(0), f(1))^t \in R^2$ Then, with the standard basis $\{1, x, x^2\}$ of P_2

$$T(a + bx + cx^2) = (a, a + b + c)^t \Rightarrow A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

4.1 Invertibility is equivalent to infectivity and subjectivity

Definition Injective (one-to-one) $N(T) = 0$, i.e., $Tv_1 = Tv_2$ implies $v_1 = v_2$.

EXAMPLE The differentiation map $T_1 = D$ is not injective since

$$D(c + p) = Dp \text{ for all constants } c \in R.$$

Integral operator $T_2 f = \int_0^x f\,dx$ is injective since

$$T_2 f = T_2 g \to f = DT_2 f = DT_2 g = g.$$

<u>**Definition** T **is surjective (onto)**</u> $R(T) = W$. For all $w \in W$, there exits $v \in V$ such that $w = Tv$.

<u>EXAMPLE</u> The differentiation map $D : P_5 \to P_5$ is not surjective, because the polynomial x^5 is not in the range of D. However, the differentiation map $D : P_5 \to P_4$ is surjective.

<u>**Theorem**</u> A linear map T is invertible if and only if it is injective and surjective.

Proof Suppose T is injective and surjective. We want to prove that T is invertible. For each $w \in W$, define Sw to be the unique element of V such that $TSw = w$ (the existence and uniqueness of such an element follow from the surjectivity and infectivity of T). Clearly TS equals the identity map on W. To prove that ST equals the identity map on V, Let $v \in V$. Then

$$T(STv) = (TS)(Tv) = I(Tv) = Tv.$$

This equation implies that $STv = v$ (because T is injective). Thus ST equals the identity map on V. To complete the proof, we need to show that S is linear. To do this, suppose $w_1,\ w_2 \in W$. Then

$$T(Sw_1 + Sw_2) = T(Sw_1) + T(Sw_2) = w_1 + w_2$$

Thus, $Sw_1 + Sw_2$ is unique element of V that T maps to $w_1 + w_2$. By the definition of S, this implies that $S(w_1 + w_2) = w_1 + w_2$. Hence S satisfies the additive property. Also, If $w \in W$ and $c \in R$

$$T(cSw) = cT(Sw) = c\,w.$$

Thus, $S(cw) = cSw$. Hence S is linear. □

4.2 Fundamental Theorem of Linear Maps

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, V)$. Then range of T is finite-dimensional and

$$\dim(V) = \dim(N(T)) + \dim(R(T)).$$

Proof: Let $\{\vec{u}_1, \cdots, \vec{u}_m\}$ be a basis of $N(T)$ and $\dim(N(T)) = m$ The linearly independent list $\{\vec{u}_1, \cdots, \vec{u}_m\}$ can be extended to a basis

$$\{\vec{u}_1, \cdots, \vec{u}_m, \vec{v}_1, \cdots, \vec{v}_n\}$$

of V. Thus $\dim(V) = m + n$. To complete the proof, we need show that $\dim(R(T)) = n$. We will do this by proving that $\{T\vec{v}_1, \cdots, T\vec{v}_n\}$ is a basis of $R(T)$. Let $\vec{v} \in V$ and

$$\vec{v} = a_1\vec{u}_1 + \cdots a_m\vec{u}_m + b_1\vec{v}_1 + \cdots b_n\vec{v}_n$$

Applying T to both sides of this equation, we get

$$T\vec{v} = a_1T\vec{u}_1 + \cdots a_mT\vec{u}_m + b_1T\vec{v}_1 + \cdots b_nT\vec{v}_n$$

Since the terms of the form $T\vec{u}_j = 0$, this implies $\{T\vec{v}_1, \cdots, T\vec{v}_n\}$ spans range of T. If we prove that $\{T\vec{v}_1, \cdots, T\vec{v}_n\}$ are linearly independent, the proof is completed. Suppose $c_1, \cdots, c_n$ and

$$c_1T\vec{v}_1 + \cdots + c_nT\vec{v}_n = 0$$

Then,

$$T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = 0$$

and $c_1\vec{v}_1 + \cdots c_n\vec{v}_n \in N(T)$. Since $\{\vec{u}_1, \cdots, \vec{u}_m\}$ be a basis of $N(T)$

$$c_1\vec{v}_1 + \cdots c_n\vec{v}_n = d_1\vec{u}_1 + \cdots d_m\vec{u}_m$$

Since $\{\vec{u}_1, \cdots, \vec{u}_m, \vec{v}_1, \cdots, \vec{v}_n\}$ are linearly independent, $c_1 = \cdots = c_m = 0$. $\square$.

Corollary 1 A map to a smaller dimensional space is not injective since $\dim(N(T)) \geq 1$.

Corollary 2 A map to a larger dimensional space is not surjective since $\dim(R(T)) < \dim(V)$.

Corollary 3 $A \in R^{n\times n}$. Then A is injective if an only if A is surjective. If so, A is nonsingular.

4.3 Change of Basis and Similarity transform

In linear algebra, two $n \times n$ matrices A and B are called similar if there exists a nonsingular $n \times n$ matrix P such that

$$B = P^{-1}AP.$$

Similar matrices represent the same linear map under two (possibly) different bases, with P being the change of basis matrix, i.e.,

$$y = Px \Leftrightarrow x = P^{-1}y.$$

A transformation

$$A \to P^{-1}AP$$

is called a similarity transformation or conjugation of the matrix A. In the general linear group, similarity is therefore the same as conjugacy, and similar matrices are also called conjugate.

Theorem (**Change of Basis**) Let $E = \{\vec{v}_1, \cdots, \vec{v}_n\}$ and $F = \{\vec{w}_1, ..., \vec{w}_n\}$ be two ordered bases for a vector space V , and let $T : V \to V$ be a linear operator Let P be the transition matrix representing the change from F to E. If A is the matrix representing T with respect to E, and B is the matrix representing T with respect to F, then $B = P^{-1}AP$. ($A : E \to E$ and $B : F \to F$ are matrix representation of $T : V \to V$)

Proof: Let x be any vector in W and let

$$\vec{w} = x_1\,\vec{w}_1 + \cdots + x_n\,\vec{w}_n$$

Let $y = Px$, $t = Ay$ and $z = Bx$. It follows from the definition of P that $y = \vec{v}|_E$, i.e.,

$$\vec{v} = y_1\,\vec{v}_1 + \cdots + y_n\,\vec{v}_n$$

Since A represents T with respect to E, and B represents T with respect to F, we have $t = T(\vec{v})|_E$ and $z = T(\vec{v})|_F$. Since the transition matrix from E to F is P^{-1} we have

$$P^{-1}APx = P^{-1}Ay = P^{-1}t = z = Bx,$$

which implies

$$P^{-1}APx = Bx \text{ for all } x \in R^n.$$

EXAMPLE Let $E = \{1, x, x^2\}$ be the standard basis to P_2. Another basis for P_2 is $F = \{x+1, x-1, 2x^2\}$. Since the transformation matrix from F to E is $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and thus the transformation matrix P from E to F is

$$\begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1}$$

Consider the element $f = a + bx + cx^2 \in P_2$. This represents the fact that f can also be written as $\frac{a+b}{2}(x+1) + \frac{b-a}{2}(x-1) + \frac{c}{2}(2x^2)$.

When defining a linear transformation, it can be the case that a change of basis can result in a simpler form of the same transformation. For example, the matrix representing a rotation in R^3 when the axis of rotation is not aligned with the coordinate axis can be complicated to compute. If the axis of rotation were aligned with the positive z-axis, then it would simply be

$$P = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where θ is the angle of rotation.

5 Eigenvalues

LEARNING OBJECTIVES FOR THIS CHAPTER invariant subspaces, eigenvalues, eigenvectors, and eigenspaces, diagonarization and Jordan form, solution to linear ordinary differential equations. Markov Chain transition matrix.

Definition (Invariant subspace) Suppose $T \in \mathcal{L}(V, V)$. A subspace U of V is called invariant under T if $u \in U$ implies $Tu \in U$.

The null space and range space of a linear transformation, are prominent examples of invariant subspaces. More importantly, a specific case of the invariant subspace is as follows. **An eigenvalue $\lambda \in C$ of an $n \times n$ matrix A** satisfies

$$(\lambda I - A)\vec{v} = 0 \Leftrightarrow A\vec{v} = \lambda\,\vec{v} \Leftrightarrow \text{ span}(\vec{v}) \text{ is an invariant subspace of } A.$$

for a nontrivial vector $\vec{v} \in C^n$, i.e., such a $\vec{v}$ is called **an eigenvector corresponding to an eigenvalue $\lambda \in C$**. Let A be an $n \times n$ matrix and $\lambda \in C$. The following statements are equivalent:

(a) λ is an eigenvalue of A.
(b) $(A - \lambda I)x = 0$ has a nontrivial solution.
(c) $N(A - \lambda I) \neq \{0\}$
(d) $(A - \lambda I)$ is singular.

(e) $\det(A - \lambda I) = 0$.
Thus, λ satisfies the characteristic equation

$$\chi(\lambda) = \det(\lambda\, I - A) = 0.$$

and there exist n eigenvalues $\{\lambda_i\}$ (including algebraic multiplicities) of A. Complex eigenvalues λ of $A \in R^{n\times n}$ appear in complex conjugate pair $\lambda = \alpha \pm i\beta$. Thus,

$$\chi(\lambda) = (\lambda - \lambda_1) \times \cdots \times (\lambda - \lambda_n)$$

EXAMPLE Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^{2\times 2}$. Then,

$$\det(A - \lambda\, I) = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc = 0$$

and eigenvalues λ are given by

$$\lambda = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

Theorem Let A and B be $n \times n$ matrices. If B is similar to A, then the two matrices have the same characteristic polynomial and, consequently, the same eigenvalues.
Proof: $B = P^{-1}AP$ and

$$\det(B - \lambda I) = \det(P(A - \lambda I)P^{-1}) = \det(P)\det(A - \lambda I)\det(P^{-1}) = \det(A - \lambda I).$$

Theorem $\lambda_1 \times \lambda_2 \times \cdots \times \lambda_n = \chi(0) = \det(A)$ and

$$\lambda_1 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn} = \text{trace}(A)\text{=sum of the diagonal entries } A,$$

which is the coefficient of $(-\lambda)^{n-1}$ of $\chi(\lambda)$.

Definition (Diagnosable) $A \in R^{n\times n}$ is said to be diagonalizable if there exists a nonsingular matrix P and a diagonal matrix Λ such that $P^{-1}AP = \Lambda$. We say that P diagonalizes A.

That is, if A is similar to a diagonal matrix Λ:

$$P^{-1}AP = \Lambda = diag(\lambda_1, \cdots, \lambda_n),$$

then λ_i are n eigenvalues of A and each column vector $\vec{v}_i$ of P is an eigenvector corresponding to λ_i, i.e., $A\vec{v}_i = \lambda_i\vec{v}_i$. Even, if A has a repeated eigenvalue λ with algebraic multiplicity $r > 1$, A has linear independent r eigenvectors corresponding to λ.

Theorem If eigenvalues $\{\lambda_i\}$ of A are distinct, there exist corresponding eigenvectors $\{\vec{v}_1, \cdots \vec{v}_n\}$ are linear independent and A is diagnosable.

Proof We prove this by induction on r. It is true for $r = 1$, because eigenvectors are non-zero by definition. For $r > 1$, suppose that for some $a_1, \cdots, a_r$ we assume

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_r\vec{v}_r = 0.$$

Then, applying A to this equation gives

$$a_1\lambda_1\vec{v}_1 + \cdots + a_r\lambda_r\vec{v}_r = 0.$$

Now, subtracting λ_1 times the first equation from the second gives

$$a_2(\lambda_2 - \lambda_1)\vec{v}_2 + \cdots + a_r(\lambda_r - \lambda_1)\vec{v}_r = 0.$$

By the inductive hypothesis, $\{\vec{v}_2, ..., \vec{v}_r\}$ are linearly independent, so $a_k(\lambda_k - \lambda_1) = 0$ and thus $a_k = 0,\ k > 1$ and also $a_1 = 0$. Thus, $\{\vec{v}_1, ..., \vec{v}_r\}$ are linearly independent. $\square$

If we let P be matrix whose column vectors consist of eigenvectors $\{v_i\}$:

$$P = [\vec{v}_1|\cdots|\vec{v}_n],$$

which are linearly independent, then $A\vec{v}_i = \lambda_i\,\vec{v}_i,\ 1 \le i \le n$ is written as a matrix identity

$$AP = P\Lambda, \quad \Lambda = diag(\lambda_1, \cdots, \lambda_n) \Leftrightarrow P^{-1}AP = \Lambda.$$

That is, A is similar to a diagonal matrix Λ $\square$

Remark (1) A is diagonalizable does not mean A has distinct eigenvalues. For example, $A = I_2$ has a repeated eigenvalue $\lambda = 1$. But A is diagonal.

(2) In general $A \in R^{n\times n}$ need not be diagonalizable Consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The characteristic polynomial is $\chi(\lambda) = (\lambda - 1)^2$, so there is a repeated eigenvalue $\lambda = 1$. The eigenvector equations

$$(A - I)\vec{v} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\vec{v} = 0$$

has a single solution $\vec{v}_1 = c\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ where c is arbitrary. That is, A is not diagnosable. To proceed we will introduce the generalized eigenvectors so that one can complete the similarity P to **a Jordan canonical form**.

(3) **Real 2×2 canonical form for complex conjugate eigenvalue case**. Assume $A \in R^{2\times 2}$ has a complex conjugate eigenvalue $\lambda = a \pm i\,b$. Let $\vec{v} = \vec{v}_1 + i\,\vec{v}_2$ be a corresponding eigenvector,

$$A\vec{v} = A\vec{v}_1 + i\,A\vec{v}_2 = \lambda\,\vec{v} = a\vec{v}_1 - b\vec{v}_2 + i\,b\vec{v}_1 + a\vec{v}_2$$

and thus equating real part and imaginary part,

$$A\vec{v}_1 = a\vec{v}_1 - b\vec{v}_2,\ A\vec{v}_2 = b\vec{v}_1 + a\vec{v}_2$$

Equivalently, if we let $P = [\vec{v}_1, \vec{v}_2]$ we have

$$P^{-1}AP = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \text{ real canonical form.}$$

EXAMPLE Consider $A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$. Then, we have

$$(1-\lambda)(3-\lambda)+2 = \lambda^2 - 4\lambda + 1 = 0 \Rightarrow \lambda = 2 \pm i.$$

and

$$\begin{pmatrix} 1-(2\pm i) & 1 \\ -2 & 3-(2\pm i) \end{pmatrix} \vec{v} = 0 \Rightarrow \vec{v} = c \begin{pmatrix} 1 \\ 1 \pm i \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

5.1 Application to ODE

Given a matrix $A \in R^{n\times n}$ consider the linear system of ordinary differential equations

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t), \quad \vec{x}(0) = \vec{x}_0$$

For an eigenpair $(\lambda, \vec{v})$

$$x(t) = ce^{\lambda t}\vec{v} \text{ with } c \text{ is a constant,}$$

is a solution to $\frac{d}{dt}x(t) = Ax(t)$, i.e.,

$$\frac{d}{dt}x(t) = c\lambda e^{\lambda t}\vec{v} = Ax(t)$$

If A is diagnosable, then $\{\vec{v}_1, \cdots, \vec{v}_n\}$ are linearly independent and there exist unique $(c_1, \cdots, c_n)$ such that

$$x_0 = c_1\vec{p}_1 + \cdots + c_n\vec{p}_n$$

and thus

$$x(t) = c_1 e^{\lambda_1 t}\vec{p}_1 + \cdots c_n e^{\lambda_n t}\vec{p}_n \quad \text{(Superposition Principle).}$$

Equivalently, $x(t) = Pe^{\Lambda t}P^{-1}x_0$, where P defines a change of basis of R^n.

EXAMPLE 1 Consider the 2×2 system

$$\frac{d}{dt}\vec{x}(t) = \begin{pmatrix} -8 & -5 \\ 10 & 7 \end{pmatrix} \vec{x}(t)$$

The characteristic polynomial is $\chi(\lambda) = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$, so there are two eigenvalues, each with algebraic multiplicity one, $\lambda_1 = 2$ and $\lambda_2 = -3$. The eigenvector equations for $\vec{p}_1$, $\vec{p}_2$ are

$$(A - 2\,I)\vec{p}_1 = \begin{pmatrix} -10 & -5 \\ 10 & 5 \end{pmatrix} \vec{p}_1 = 0 \Rightarrow \vec{p}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$(A + 3\,I)\vec{p}_2 = \begin{pmatrix} -5 & -5 \\ 10 & 10 \end{pmatrix} \vec{p}_2 = 0 \Rightarrow \vec{p}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Thus, we have

$$x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

EXAMPLE 2 Consider

$$\frac{d}{dt}x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x(t)$$

The characteristic polynomial is $\chi(\lambda) = (\lambda - 1)^2$, so there is a repeated eigenvalue $\lambda = 1$. The eigenvector equations

$$(A - I)\vec{v} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \vec{v} = 0$$

has a single solution $\vec{v}_1 = c \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ where c is arbitrary. One needs to find second one. Using this eigenvector, we compute the generalized eigenvector $\vec{v}_2$ by solving

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1.$$

Writing out the values:

$$\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This gives

$$\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, if $P = [\vec{v}_1 | \vec{v}_2]$ then

$$AP = P \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Note that

$$\vec{v}_1 = (A - \lambda I)\vec{v}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$(A - \lambda I)^2 \vec{v}_2 = (A - \lambda I)\vec{v}_1 = \vec{0} \text{ and } (A - \lambda I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Also, we have

$$x(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2(1 + t)e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

since $((1 + t)e^t)' = (1 + t)e^t + e^t$.

EXAMPLE 3 This example is more complex than Example 1. A upper triangular matrix A:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 6 & 3 & 2 & 0 & 0 \\ 10 & 6 & 3 & 2 & 0 \\ 15 & 10 & 6 & 3 & 2 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ since $\chi(\lambda) = (\lambda - 1)^2(\lambda - 2)^3 = 0$ with algebraic multiplicities 2 and 3, respectively. The generalized eigenspaces of A are calculated below:

x_1 is the ordinary eigenvector associated with $\lambda_1 = 1$ and x_2 s a generalized eigenvector associated with $\lambda_2 = 2$. y_1 is the ordinary eigenvector associated with $\lambda_2 = 2$, y_2, y_3 are generalized eigenvectors associated with λ_2.

$$(A - 1\,I)\mathbf{x}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 \\ 15 & 10 & 6 & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -9 \\ 9 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0},$$

$$(A - 1\,I)\mathbf{x}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 10 & 6 & 3 & 1 & 0 \\ 15 & 10 & 6 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -15 \\ 30 \\ -1 \\ -45 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -9 \\ 9 \\ -3 \end{pmatrix} = \mathbf{x}_1,$$

$$(A - 2\,I)\mathbf{y}_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 & 0 \\ 10 & 6 & 3 & 0 & 0 \\ 15 & 10 & 6 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0},$$

$$(A - 2\,I)\mathbf{y}_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 & 0 \\ 10 & 6 & 3 & 0 & 0 \\ 15 & 10 & 6 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 9 \end{pmatrix} = \mathbf{y}_1,$$

$$(A - 2\,I)\mathbf{y}_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 & 0 \\ 10 & 6 & 3 & 0 & 0 \\ 15 & 10 & 6 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} = \mathbf{y}_2.$$

This results in a basis for each of the generalized eigenspaces of A. Together the two chains of generalized eigenvectors span the space of all 5-dimensional column vectors.

$$\{\mathbf{x}_1, \mathbf{x}_2\} = \left\{ \begin{pmatrix} 0 \\ 3 \\ -9 \\ 9 \\ -3 \end{pmatrix} \begin{pmatrix} 1 \\ -15 \\ 30 \\ -1 \\ -45 \end{pmatrix} \right\}, \{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 9 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 0 \end{pmatrix} \right\}.$$

An "almost diagonal" matrix J in Jordan normal form, similar to A is obtained as follows:

$$P = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 3 & -15 & 0 & 0 & 0 \\ -9 & 30 & 0 & 0 & 1 \\ 9 & -1 & 0 & 3 & -2 \\ -3 & -45 & 9 & 0 & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

where P is a generalized eigen matrix for A, the columns of P are a canonical basis for A, and $AP = PJ$.

In general A is diagnosable if and only if the sum of the dimensions of the eigenspaces is n. Or, equivalently, if and only if A has n linearly independent eigenvectors. Not all matrices are diagonalizable; matrices that are not diagonalizable are called defective matrices. In addition to the above examples consider the following matrix:

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}.$$

Including multiplicity, the eigenvalues of A are $\lambda = 1, 2, 4, 4$. The dimension of the eigenspace corresponding to the eigenvalue $\lambda = 4$ is 1 (and not 2), so A is not diagonalizable. However, there is an invertible matrix P such that $J = P^{-1}AP$, where

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

The matrix J is almost diagonal. This is the Jordan normal form of A. For $i = 1, 2, 3$ there exits a eigenvector $p_i \in N(\lambda_i I - A)$. For a repeated (algebraic) eigenvalue $\lambda_3 = \lambda_4 = 4$ $(4\,I - A)$ does not have two independent eigenvectors. But, there exists $p_4 \in N(\lambda_4 I - A)^2)$ satisfying

$$(\lambda_4 I - A)p_4 = p_3,$$

where p_3 is an eigenvector of A corresponding to $\lambda_3 = \lambda_4 = 4$. p_4 are called a generalized eigenvector of A.

<u>Exercise</u> Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$.

Solution:

$$\det \begin{pmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{pmatrix} = (2-\lambda)((-2-\lambda)(2-\lambda)+3)+3(2-\lambda)-3-3-(-2-\lambda) = -\lambda(\lambda^2-2\lambda+1) = 0$$

Thus, the eigenvalues are $\lambda_1 = 0, \lambda_2 = \lambda_3 = 1$. For $\lambda = 0$

$$\begin{pmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{pmatrix} \vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda = 1$

$$\begin{pmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{pmatrix} \vec{v} = \begin{pmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{v} = 0,$$

which implies that $\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Thus, A is diagonizable.

5.2 Reduced form

In general, a square complex matrix A is similar to a block diagonal matrix

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}$$

where each block J_i is a square matrix of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}.$$

Definition (Symmetric) $A \in R^{n \times n}$ is said to be symmetric if $A^t = A$.

Definition (Real orthogonal) $A \in R^{n \times n}$ is said to be orthogonal if $A^t = A^{-1}$, or equivalently, if $AA^t = A^t A = I_n$.

Theorem (Symmetric Case) If $A \in R^{n \times n}$ is a symmetric matrix, then there exits a real orthogonal matrices U ($U^t U = I$) such that

$$AU = \Lambda U \Leftrightarrow A = U \Lambda U^t$$

where $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$ with $\lambda_i \in R$ is i-th eigenvalue of A and i-th column vector of U is the corresponding eigenvector to λ_i.

Proof: An eigenvalue λ of a matrix A is characterized by the algebraic relation $Mu = \lambda\, u$. When A is $n \times n$ symmetric matrix, a variational characterization (Riesz method) is also available. Consider a constrained maximization

$$f(x) = x^t Ax \text{ subject to } |x|^2 \leq 1$$

By Weierstrass theorem. there exist a maximizer u and by the Lagrange multipliers theorem $L(x, \lambda) = f(x) + \lambda\,(1 - |x|^2)$ satisfies

$$\frac{1}{2} L_x(u, \lambda) = Au - \lambda u = 0 \text{ and } |u|^2 = 1$$

Therefore $Au = \lambda\, u$ and $|u| = 1$. For every unit length eigenvector u of A its eigenvalue is $f(u)$, so λ is the largest eigenvalue of A. The same calculation performed on the orthogonal complement of u, i.e., $\{x \in R^n : (x, u) = 0\}$ gives the next largest eigenvalue of A, and so. That is, we obtain eigen pairs (λ_i, u_i) such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\{u_i\}$ is orthonormal i.e. $(u_i, u_j) = \delta_{i,j}$. $\square$

Theorem (Jordan form $A = PJP^{-1}$) Given an eigenvalue λ, its corresponding Jordan block gives rise to a Jordan chain. The generator, or lead vector, say p_r, of the chain is a generalized eigenvector such that $(A - \lambda\, I)^r p_r = 0$, where r is the size of the Jordan block. The vector $p_1 = (A - \lambda I)^{r-1} p_r$ is an eigenvector corresponding to λ. In general, p_i is a preimage of p_{i-1} under $A - \lambda I$, i.e., $(A - \lambda\, I)p_i = p_{i-1}$. So the lead vector generates the chain via multiplication by $(A - \lambda\, I)$. Thus, $AP = PJ_i$ for each Jordan chain. Therefore, the statement that every square matrix A can be put in Jordan normal form is equivalent to the claim that there exists a basis consisting only of eigenvectors and generalized eigenvectors of A.

5.2.1 Matrix exponential solution

Recall: Given a eigenvalue λ we have

$$e^{\lambda t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda^k$$

In the case of matrix A the matrix exponential is easy to compute:

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

and

$$x(t) = e^{At} x(0)$$

defines the solution to the differential equation. If $A = P^{-1}BP$, then

$$e^{At} = P^{-1} e^{Bt} P$$

For example, B is a diagonal matrix Λ

$$e^{\Lambda t} = diag(e^{\lambda_1 t}, \cdots, e^{\lambda_n t}).$$

If B is a Jordan block J of size r, we have

$$exp(Jt) = exp(\lambda t)(I + Nt + \cdots + \frac{t^{r-1}}{(r-1)!} N^{r-1})$$

and for $B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

$$exp(Bt) = \begin{bmatrix} \cos(at) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}.$$

Moreover, Let $f(z)$ be an analytical function of a complex argument. Applying the function on a $n \times n$ Jordan block J with eigenvalue λ results in an upper triangular matrix:

$$f(J) = \begin{bmatrix} f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2} & \dots & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ 0 & f(\lambda) & f'(\lambda) & \dots & \frac{f^{(n-2)}(\lambda)}{(n-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & f(\lambda) & f'(\lambda) \\ 0 & 0 & 0 & 0 & f(\lambda) \end{bmatrix},$$

so that the elements of the k-th super-diagonal of the resulting matrix are $\frac{f^{(k)}(\lambda)}{k!}$. For a matrix of general Jordan normal form the above expression shall be applied to each Jordan block. The following example shows the application to the power function $f(z) = z^n$:

$$\begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}^n = \begin{bmatrix} \lambda_1^n & \binom{n}{1}\lambda_1^{n-1} & \binom{n}{2}\lambda_1^{n-2} & 0 & 0 \\ 0 & \lambda_1^n & \binom{n}{1}\lambda_1^{n-1} & 0 & 0 \\ 0 & 0 & \lambda_1^n & 0 & 0 \\ 0 & 0 & 0 & \lambda_2^n & \binom{n}{1}\lambda_2^{n-1} \\ 0 & 0 & 0 & 0 & \lambda_2^n \end{bmatrix},$$

where the binomial coefficients are defined as $\binom{n}{k} = \prod_{i=1}^{k} \frac{n+1-i}{i}$. For integer positive n it reduces to standard definition of the coefficients. For negative n the identity $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$ may be of use.

Real Jordan Form decomposition $A = PJP^{-1}$. The real Jordan block is given by

$$J_i = \begin{bmatrix} C_i & I & & \\ & C_i & \ddots & \\ & & \ddots & I \\ & & & C_i \end{bmatrix}.$$

where for non-real eigenvalue $a_i + ib_i$ with given algebraic multiplicity of the 2×2 matrix form

$$C_i = \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix}.$$

This real Jordan form is a consequence of the complex Jordan form. For a real matrix the nonreal eigenvectors and generalized eigenvectors can always be chosen to form complex conjugate pairs. Taking the real and imaginary part (linear combination of the vector and its conjugate), the matrix has this form with respect to the new basis.

Real Schur decomposition For $A \in R^{n \times n}$ one can always write $A = USU^t$ where $U \in R^{n \times n}$ is a real orthogonal matrix, $U^tU = I_n$, S is a block upper triangular matrix called the real Schur form. The blocks on the diagonal of S are of size 1×1 (in which case they represent real eigenvalues) or 2×2 (in which case they are derived from complex conjugate eigenvalue pairs). QR-algorithm is used to obtain S and U.

Basic QR-algorithm Let $A_0 = A$. At the k-th step (starting with $k = 0$), we compute the QR decomposition $A_k = Q_k R_k$ We then form $A_{k+1} = R_k Q_k$. Note that

$$A_{k+1} = R_k Q_k = Q_k^{-1} A_k Q_k = Q_k^t A_k Q_k,$$

so all the A_k are similar to A and hence they have the same eigenvalues. The algorithm is numerically stable because it proceeds by orthogonal similarity transforms. Let

$$\hat{Q}_k = Q_0 \cdots Q_k \text{ and } \hat{R}_k = R_0 \cdots R_k$$

be the orthogonal and triangular matrices generated by the QR algorithm, Then, we have

$$A_{k+1} = \hat{Q}_k^t A \hat{Q}_k$$

With shifts $\sigma_0, \cdots, \sigma_k$, starting with A. Then

$$\hat{Q}_k \hat{R}_k = (A - \sigma_0\, I) \cdots (A - \sigma_k\, I),$$

which is used to prove the convergence of (Q_k, R_k) to (U, S)

5.3 Markov Chain Transition matrix

Suppose there is a physical or mathematical system that has k possible states and at any one time, the system is in one and only one of its k states. And suppose that at a given observation period, say n^{th} period, the probability of the system being in a particular state depends on its status at the $n-1$ period, such a system is called Markov Chain or Markov process. Define a_{ij} to be the probability of the system to be in state i after it was in state j (at any observation). The matrix $A^t = (a_{ji})$ is called the Transition matrix of the Markov Chain.

EAMPLE In a certain town, 30% of the married women get divorced each year and 20 % of the single women get married each year. $A = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$. Eigenvalues λ satisfy

$$(.7 - \lambda)(.8 - \lambda) - .06 = \lambda^2 - 1.5\lambda + .5 = (\lambda - 1)(\lambda - .5) = 0$$

Eigenvector for $\lambda_1 = 1$: $(A - \lambda I)\vec{v}_1 = 0$ where

$$A - \lambda I = \begin{pmatrix} 0.7 - 1 & 0.2 \\ 0.3 & 0.8 - 1 \end{pmatrix} = \begin{pmatrix} -.03 & 0.2 \\ 0.3 & -0.2 \end{pmatrix}$$

and thus $\vec{v}_1 = c \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Eigenvector for $\lambda_2 = .5$: $(A - \lambda I)\vec{v}_2 = 0$ where

$$A - \lambda I = \begin{pmatrix} 0.7 - 0.5 & 0.2 \\ 0.3 & 0.8 - 0.5 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & -0.3 \end{pmatrix}$$

and thus $\vec{v}_2 = c \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

EAMPLE Suppose in small town there are three places to eat, two restaurants one Chinese and another one is Mexican restaurant. The third place is a pizza place. Everyone in town eats dinner in one of these places or has dinner at home. Assume that 20% of those who eat in Chinese restaurant go to Mexican next time, 20% eat at home, and 30% go to pizza place. From those who eat in Mexican restaurant, 10% go to pizza place, 25% go to Chinese restaurant, and 25% eats at home next time. From those who eat at pizza place 30% Those who eat at home 20% go to Chinese, 25% go to Mexican place, and 30% to pizza place. We call this situation a system. A person in the town can eat dinner in one of these four places, each of them called a state. In our example, the system has four states. We are interested in success of these places in terms of their business. So, transition matrix for this example above, is

$$A = \begin{pmatrix} .25 & .20 & .25 & .30 \\ .20 & .30 & .25 & .30 \\ .25 & .20 & .40 & .10 \\ .30 & .30 & .10 & .30 \end{pmatrix}$$

Note that the sum of each column in this matrix is one. Any matrix with this property is called a (left) stochastic matrix, probability matrix or a Markov matrix. Define a (column) state vector $\vec{x}$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix},$$

where, x_i = probability that the system is in the i^{th} state at the time of observation. That is, $\vec{x}$ is a probability vector, i.e., $x_i \geq 0$ and the sum of the entries of the state vector has to be one:

$$x_1 + x_2 + \cdots + x_k = 1.$$

Question What is the probability that the system is in the i^{th} state, at the n^{th} observation?
Answer: $x^{(n)} = A^n \vec{x}^{(0)}$ where $\vec{x}^{(0)}$ is an initial probability vector.

For example, if $x^{(0)} = (1, 0, 0, 0)^t$ we have

$$x^{(5)} = \begin{pmatrix} .2495 \\ .2634 \\ .2339 \\ .2532 \end{pmatrix}, \; x^{(10)} = \begin{pmatrix} .2495 \\ .2634 \\ .2339 \\ .2532 \end{pmatrix}, \; x^{(20)} = \begin{pmatrix} .2495 \\ .2634 \\ .2339 \\ .2532 \end{pmatrix}.$$

This suggests that the state vector approached to some fixed vector, as the number of observation periods increase. In fact, the eigenvalues of A are $1.0000, -0.0962, 0.0774, 0.2688$ and the first eigen state is $(0.2495, 0.2634, 0.2339, 0.2532)$, which is the asymptotic probability vector $\lim_{n\to\infty} \vec{x}^{(n)}$, independent of $\vec{x}^{(0)}$.

This is not the case for every Markov Chain. For example, if

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem If a Markov chain with an $n \times n$ transition matrix A converges to a steady-state vector x, then

(i) x is a probability vector.

(ii) $\lambda_1 = 1$ is an eigenvalue of A and x is an eigenvector corresponding to $\lambda = 1$.

(iii) If $\lambda_1 = 1$ is a dominant eigenvalue of a (left) stochastic matrix A (i.e., $|\lambda_i| < 1,\ i \geq 2$) , then the Markov chain with transition A will converge to a steady-state vector.

Proof: Since $\sum_{i=1}^k a_{i,j}$ for all j, $\lambda_1 = 1$ is an eigenvalue. Next, if x is a probability vector, so is $y = Ax$ since

$$\sum_{i=1}^k y_i = \sum_{i=1}^k \sum_{j=1}^k a_{i,j} x_j = \sum_{j=1}^k (\sum_{i=1}^k a_{i,j}) x_j = \sum_{j=1}^k x_j = 1$$

For any probability vector $\vec{x}^{(0)}$

$$\vec{x}^{(0)} = a_1 \vec{v}_1 + \cdots a_k \vec{v}_k \ \ \text{(assuming } A \text{ is diagnosable)}$$

and

$$A^n \vec{x}^0 = a_1 \vec{v}_1 + a_2 (\lambda_2)^n + \cdots a_k (\lambda_k)^n \vec{v}_i \to a_1 \vec{v}_1. \Box$$

6 Inner product and Orthogonality

LEARNING OBJECTIVES FOR THIS CHAPTER: Inner product and Orthogonality of vectors, Gram-Schmidt orthogonalization, Orthogonal decomposition theorem and Minimum norm solution, Least Square solution to linear system of equations, Generalized matrix inverse.

The dot (inner) product of two vectors $x = (x_1, \cdots, x_n)^t$ and $y = (y_1, \cdots, y_n)^t$

$$x \cdot y = (x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Note that for $c \in R$ and $x,\ y,\ z \in R^n$

$$(y, x) = (x, y), \ \ (cx, y) = c(x, y), \ \ (x + y, z) = (x, z) + (y, z)$$

For example, in $x = (1, 3, -5),\ y = (4, -2, -1) \in R^3$

$$(x, y) = (1 \times 4) + (3 \times -2) + (-5 \times -1) = 4 - 6 + 5 = 3.$$

Transpose of matrix A^t For $A \in R^{m \times n}$, $(Ax, y)_{R^m} = (x, A^t y)_{R^n}$ since

$$(Ax, y)_{R^m} = \sum_{i=1}^m (\sum_{j=1}^n a_{ij} x_j) y_i = \sum_{j=1}^n (\sum_{i=1}^n a_{ij} y_i) x_j = (x, A^t y)_{R^n}.$$

Inner product space An inner product space is a vector space V over the field $F = R$ together with a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

called an inner product that satisfies the following conditions for all vectors $x, y, z \in V$ and all scalars a:

$$\langle ax, y\rangle = a\langle x, y\rangle$$
$$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

and

$$\langle x, y\rangle = \overline{\langle y, x\rangle}, \quad \langle x, x\rangle > 0 \text{ for } x \neq 0.$$

EXAMPLE $V = C(-1,1)$ and $\langle x, y\rangle = \int_{-1}^{1} x(t)y(t)\,dt$.

EXAMPLE $V = P_n$ and $\langle t^k, t^j\rangle = \int_{-1}^{1} t^k t^j\,dt = \frac{1}{k+j+1}(1-(-1)^{k+j+1})$.

Geometrically, we have the norm and cosine angle,

$$\|\vec{x}\| = \sqrt{(\vec{x}, \vec{x})} = \sqrt{x_1^2 + \cdots + x_n^2}$$

$$cos(\theta) = \frac{(\vec{x}, \vec{y})}{\|\vec{x}\|\|\vec{y}\|}$$

where $\|\vec{x}\|$ **is the norm of** $\vec{x}$ **and** θ **is the angle between vectors** $\vec{x}$ **and** $\vec{y}$. Since for all $t \in R$

$$0 \le \|\vec{x} + t\vec{y}\|^2 = (\vec{x} + t\vec{y}, \vec{x} + t\vec{y}) = (\vec{x}, \vec{x}) + 2t(x, y) + t^2(y, y) = \|\vec{x}\|^2 + 2t(x, y) + t^2\|y\|^2$$

we have the Cauchy Schwarz inequality (letting $t = -\frac{(x,y)}{\|y\|^2}$)

$$|(\vec{x}, \vec{y})| \le \|\vec{x}\|\|\vec{y}\|.$$

Thus,

$$\|\vec{x} + \vec{y}\|^2 \le \|\vec{x}\|^2 + 2(\vec{x}, \vec{y}) + \|\vec{y}\|^2 \le \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}|^2 = (\|\vec{x}\| + \|\vec{y}\|)^2$$

and we obtain the triangle inequality:

$$\|\vec{x} + \vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$$

Thus, $(R^n, \|\cdot\|)$ is a normed space ($\|x\| = 0$ iff $\vec{x} = 0$ and $\|c\vec{x}\| = |c|\|\vec{x}\|$ for all $\vec{x} \in R^n$ and $c \in R$).

In mathematics, particularly linear algebra and numerical analysis, the Gram-Schmidt process is a method for orthonormalizing a set of vectors in an inner product space, most commonly the Euclidean space R^n equipped with the standard dot product. The Gram-Schmidt process takes a finite, linearly independent set $S = \{\vec{v}_1, ..., \vec{v}_k\}$ for $k \le n$ and generates **an orthogonal set** $\tilde{S} = \{\vec{u}_1, \cdots, \vec{u}_k\}$**,** $(\vec{u}_i, \vec{u}_j) = 0,\ i \neq j$ that spans the same k-dimensional subspace S of R^n.

The method is named after Jorgen Pedersen Gram and Erhard Schmidt, but Pierre-Simon Laplace had been familiar with it before Gram and Schmidt. In the theory of Lie group decompositions it is generalized by the Iwasawa decomposition. The application of the Gram-Schmidt process to the column vectors of a full column rank matrix yields the QR decomposition (it is decomposed into an orthogonal and a triangular matrix).

We define the projection operator by

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u},$$

i.e., this operator projects the vector $\mathbf{v}$ orthogonally onto the line spanned by vector $\mathbf{u}$ since

$$\langle \mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v}), \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{u} \rangle = 0.$$

The Gram-Schmidt orthogonalization works as follows:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, & \tilde{u}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), & \tilde{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), & \tilde{u}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
\mathbf{u}_4 &= \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_3}(\mathbf{v}_4), & \tilde{u}_4 &= \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|} \\
&\vdots & &\vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k), & \tilde{u}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.
\end{aligned}$$

The sequence $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is the required system of orthogonal vectors, and the normalized vectors $\{\tilde{u}_1, \cdots, \tilde{u}_k\}$ form an orthonormal set. Equivalently,

$$A = [\vec{v}_1|\cdots|v_k] = QR \text{ with } Q = [\tilde{u}_1|\cdots|\tilde{u}_k]$$

where $Q^tQ = I$ and R is a upper triangular (coefficient) matrix.

$\{\vec{u}_j\}_{j=1}^n$ **is orthogonal basis of** V **we have the Fourier formula:**

$$\vec{u} = a_1\,\vec{u}_1 + \cdots + a_n\,\vec{u}_n \text{ with } a_j = \frac{\langle \vec{u}, \vec{u}_j \rangle}{\langle \vec{u}_j, \vec{u}_j \rangle},$$

since $(\vec{u}, \vec{u}_j) = a_j \langle \vec{u}_j, \vec{u}_j \rangle$ **and**

$$\|\vec{u}\|^2 = a_1^2\|\vec{u}_1\|^2 + \cdots a_n^2\|\vec{u}_n|^2.$$

6.1 Orthogonal decomposition

Definition (Orthogonal complement)

$$S^{\perp} = \{x \in R^n : (x, s) = 0 \text{ for all } s \in S\}$$

is the orthogonal complement of a subspace S of R^n.

Theorem (**The Orthogonal Decomposition Theorem**) Let S be a subspace of R^n. Then each $x \in R^n$ can be uniquely represented in the form

$$x = \hat{x} + z \text{ and } \|x\|^2 = \|\hat{x}\|^2 + \|z\|^2$$

where $\hat{x} \in S$ and $z \in S^\perp$ ($R^n = S \oplus S^\perp$)

Proof: Let $(\vec{u}_1, \cdots \vec{u}_n)$ be any orthonormal basis of S, i.e. for any $s \in S$

$$s = (s, \vec{u}_1)\, \vec{u}_1 + \cdots + (s, \vec{u}_n)\, \vec{u}_n$$

If for $x \in R^n$ define

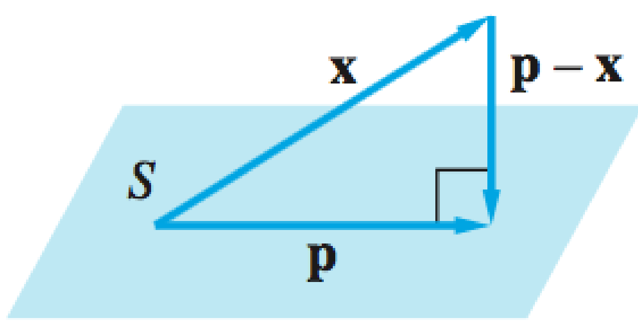
$$\hat{x} = (x, \vec{u}_1)\, \vec{u}_1 + \cdots + (x, \vec{u}_n)\, \vec{u}_n \in S$$

then $z = x - \hat{x} \in S^\perp$ since $(z, \vec{u}_i) = (x, \vec{u}_i) - (\hat{x}, \vec{u}_i) = 0$. The decomposition is unique since if there exit tow decompositions of x

$$x = \hat{x}_1 + z_1 = \hat{x_2} + z_2$$

then

$$\hat{x}_1 - \hat{x_2} = z_2 - z_1 \in S \cap S^\perp \Rightarrow \hat{x}_1 - \hat{x_2} = z_2 - z_1 = 0.$$



Orthogonal decomposition theorem II For a matrix $A \in R^{m \times n}$. Then $N(A) = R(A^t)^\perp$ and thus from the orthogonal decomposition theorem

$$R^n = N(A) \oplus R(A^t).$$

Proof: Suppose $x \in N(A)$, then

$$(x, A^t y)_{R^n} = (Ax, y)_{R^m} = 0$$

for all $y \in R^m$ and thus $N(A) \subset R(A^t)^\perp$. Conversely, $x^* \in R(A^t)^\perp$, i.e.,

$$(x^*, A^t y) = (Ax^*, y) = 0 \quad \text{for all } y \in R^m$$

Thus, $Ax^* = 0$ and $R(A^t)^\perp \subset N(A)$. $\square$

6.2 Generalized inverse

Minimum norm solution Similarly, we have

$$R^m = N(A^t) \oplus R(A), \quad R^n = N(A) \oplus R(A^t).$$

Suppose $A \in R^{m\times n},\ n \geq m$ (under-determined) wth $R(A) = R^m$, equivalently $N(A^t) = \{0\}$. From the theorem $x = A^t y$ defines the minimum solution to $Ax = b$ and thus y satisfies $AA^t y = b$ and

$$x = A^t(AA^t)^{-1}b \text{ is the minimum norm solution.}$$

Note that if $y \in N(AA^t)$ then $(y, AA^t y) = |A^t y|^2 = 0$ and $y = 0$, i.e., AA^t is nonsingular.

Least square solution Recall that we have if $N(A) = \{0\}$, equivalently $R(A^t) = R^n$ for $A \in R^{n\times n}$ so that $Ax = b$ has a unique. In general for $A \in R^{m\times n},\ m \geq n$ (over-determined),

$$x = (A^tA)^{-1}A^t b \text{ defines the least square solution.}$$

that minimizes the error $\|Ax - b\|^2$. Note that if $x \in N(A^tA)$ then $(x, A^tAx) = |Ax|^2 = 0$ and $x = 0$, i.e., A^tA is nonsingular.

Generalized inverse of A Consider the regularized least squares formulation for $\alpha > 0$

$$\min \quad J(x) = \|Ax - b\|^2 + \alpha\,\|x\|^2$$

Then the minimizer x is given by

$$x^* = (A^tA + \alpha I)^{-1}A^t b.$$

Note that if $x \in N(A^tA + \alpha I)$ then $(x, (A^tA + \alpha I)x) = |Ax|^2 + |x|^2 = 0$ and $x = 0$, i.e., $A^tA + \alpha I$ is nonsingular. In fact for $\tilde{x} \in R^n$

$$J(\tilde{x}) = \|A(\tilde{x} - x^*) - (Ax^* - b)\|^2 + \alpha\,\|\tilde{x} - x^* + x^*\|^2$$

$$= \|A(\tilde{x} - x^*)\|^2 + \|\tilde{x} - x^*\|^2 - 2(A(\tilde{x} - x^*), Ax^* - b) - 2(\tilde{x} - x^*, x^*) + J(x^*)$$

where

$$(A(\tilde{x} - x^*), Ax^* - b) + (\tilde{x} - x^*, x^*) = (\tilde{x} - x^*, (A^tA + \alpha I)x^* - A^t b) = 0.$$

Thus, we have

$$J(\tilde{x}) \geq J(x^*)$$

whose equality if and only if $\tilde{x} = x^*$.

6.3 Approximation Theory

Let S be a subspace of V. Consider the least square approximation of $\vec{u}$ in V by a linear combination of vectors in $S = \{\vec{u}_1, \cdots, \vec{u}_n$ by the least square criterion

$$\min \quad \|\vec{u} - s\|^2, \quad \text{over } s = x_1\vec{u}_1 + \cdots + x_n\vec{u}_n \in S$$

If $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is an orthonormal basis of S then it follows from the orthogonal decomposition theory

$$s^* = \langle \vec{u}, \vec{u}_1 \rangle \, \vec{u}_1 + \cdots + \langle \vec{u}, \vec{u}_n \rangle \, \vec{u}_n$$

is the best approximation of $\vec{u}$. In general,

$$\|\vec{u} - s\|^2 = |Ax - b|^2_{R^n}$$

where $A = [\vec{u}_1 | \cdots | \vec{u}_n]$. The best solution x^* is given by

$$x^* = (A^t A)^{-1}(A^t b),$$

where

$$(A^t A)_{ij} = \langle \vec{u}_i, \vec{u}_j \rangle, \quad A^t b = (\langle \vec{u}, \vec{u}_1 \rangle, \cdots, \langle \vec{u}, \vec{u}_n \rangle)^t.$$

EXAMPLE 1 (**Polynomial approximation**) Let S be the subspace P_1 of all linear functions in $C[0,1]$. Although the functions 1 and x span S, they are not orthogonal. By the Gram-Schmitz orthogonalization $u_2(x) = \sqrt{12}(x - \frac{1}{2})$ is orthogonal to $u_1 = 1$, i.e. $\{u_0(x), u_1(x)\}$ is an orthonormal basis of P_1. Thus, the best linear approximation $u(x) = e^x$ is given by $a_1 + a_2 \, u_2(x)$ with

$$\int_0^1 e^x u_1(x) \, dx = e - 1, \; a_2 = \int_0^1 e^x u_2(x) \, dx = \sqrt{3}(3 - e).$$

Next, let $S = P_3$. The, we evaluate matrix $Q_{kj} = \int_0^1 x^k x^j \, dx = \frac{1}{k+j+1}(1 - (-1)^{k+j+1})$ and vector $c_k = \int_0^1 e^x x^{k-1} \, dx, \; k = 1, 2, 3.4$. Then, the best cubic approximation is given by

$$a_1 + a_2 \, x + a_3 \, x^2 + a_4 \, x^3$$

where $a \in R^4$ solves $Qa = c$.

EXAMPLE 2 (**Fourie cosine series**) Let V be a space of even functions in $C[-\pi, \pi]$. and $S = \{\frac{1}{\sqrt{2}}, \cos(x), \cos(2x), \cdots, \cos(nx)\}$ Then, $\{\cos(k\pi x)\}_{k=0}^n$ is an orthonormal set of vectors in V, i.e., $\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(k\pi) \cos(j\pi x) \, dx = 0$ for $k \neq j$ with inner product defined by

$$\langle u, v \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) v(x) \, dx.$$

It follows from the orthogonal decomposition theory

$$s^* = \langle \vec{u}, \vec{u}_1 \rangle \, \vec{u}_1 + \cdots + \langle \vec{u}, \vec{u}_n \rangle \, \vec{u}_n$$

where the Fourier coefficient is given by

$$\langle \vec{u}, \vec{u}_k \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \cos k\pi x \, dx, \;\; k \geq 1$$

is the best approximation of a function $u(x) \in V$.

7 QR decomposition and Singular value decomposition

Householder transform and QR decomposition $A = QR$

Let $\mathbf{e}_1$ be the vector $(1, 0, \cdots, 0)^t$, $||\cdot||$ is the Euclidean norm

$$\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_m^2 = x^t x = (x, x)$$

and I is an $m \times m$ identity matrix, set

$$\mathbf{u} = \mathbf{x} - \alpha\mathbf{e}_1,$$
$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|},$$
$$Q = I - 2\mathbf{v}\mathbf{v}^\mathsf{T}.$$

Or,

$$Q = I - 2\mathbf{v}\mathbf{v}^t.$$

where Q is an $m - by - m$ Householder matrix and

$$Q\mathbf{x} = \begin{pmatrix} \alpha & 0 & \cdots & 0 \end{pmatrix}^\mathsf{T}.$$

Note that

$$Q^t Q = (I - 2\mathbf{v}\mathbf{v}^t)(I - 2\mathbf{v}\mathbf{v}^t.) = I - 4\mathbf{v}\mathbf{v}^t + 4\mathbf{v}\mathbf{v}^t = I$$

This can be used to sequentially transform an $m - by - n$ matrix A to upper triangular form. First, we multiply A with the Householder matrix Q_1 we obtain when we choose the first matrix column for x. This results in a matrix $Q_1 A$ with zeros in the left column (except for the first row).

$$Q_1 A = \begin{bmatrix} \alpha_1 & \star & \dots & \star \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{bmatrix}$$

This can be repeated for A_1 (obtained from $Q_1 A$ by deleting the first row and first column), resulting in a Householder matrix Q_2. Note that Q_2 is smaller than $Q1$. Since we want it really to operate on $Q_1 A$ instead of A we need to expand it to the upper left, filling in a 1, or in general:

$$Q_k = \begin{pmatrix} I_{k-1} & 0 \\ 0 & Q_k' \end{pmatrix}.$$

After k iterations of this process, $k = \min(m-1, n)$

$$R = Q_k \cdots Q_2 Q_1 A$$

is an upper triangular matrix. So, with

$$Q = Q_1^\mathsf{T} Q_2^\mathsf{T} \cdots Q_t^\mathsf{T},$$

$A = QR$ is a QR decomposition of A.

Remark Note that Q is a **real orthogonal transform $Q^tQ = I$ and $Q^t = Q^{-1}$** and

$$\|Q\vec{x}\| = \|\vec{x}\| \quad (\textbf{norm preserving} \text{ for all } x \in R^m).$$

In fact,

$$Q^tQ = Q_k \cdots Q_2 Q_1 Q_1^t Q_2^t \cdots Q_k^t = I$$

and

$$||Qx||^2 = (Qx)^t Qx = x^t Q^t Qx = x^t x = ||x||^2$$

QR method for Eigenvalue problems In numerical linear algebra, the QR algorithm is an eigenvalue algorithm: that is, a procedure to calculate the eigenvalues and eigenvectors of a matrix. The QR algorithm was developed in the late 1950s by John G. F. Francis and by Vera N. Kublanovskaya, working independently. The basic idea is to perform a QR decomposition, writing the matrix as a product of an orthogonal matrix and an upper triangular matrix, multiply the factors in the reverse order and iterate. It is a power method to compute dominant eigen value-pairs.
Basic QR-algorithm Let $A_0 = A$. At the k-th step (starting with $k = 0$), we compute the QR decomposition $A_k = Q_k R_k$ We then form $A_{k+1} = R_k Q_k$. Note that

$$A_{k+1} = R_k Q_k = Q_k^{-1} A_k Q_k = Q_k^t A_k Q_k,$$

so all the A_k are similar to A and hence they have the same eigenvalues. The algorithm is numerically stable because it proceeds by orthogonal similarity transforms. Let

$$\hat{Q}_k = Q_0 \cdots Q_k \text{ and } \hat{R}_k = R_0 \cdots R_k$$

be the orthogonal and triangular matrices generated by the QR algorithm, Then, we have

$$A_{k+1} = \hat{Q}_k^t A \hat{Q}_k$$

With shifts $\sigma_0, \cdots, \sigma_k$, starting with A. Then

$$\hat{Q}_k \hat{R}_k = (A - \sigma_0 I) \cdots (A - \sigma_k I)$$

7.1 PCA(Principal Component Analysis

PCA is defined as an orthogonal linear transformation that transforms the data to a new coordinate system such that the greatest variance by some scalar projection of the data comes to lie on the first coordinate (called the first principal component), the second greatest variance on the second coordinate, and so on.

Consider an $n \times p$ data matrix, X, with column-wise zero empirical mean (the sample mean of each column has been shifted to zero), where each of the n rows represents a different repetition of the experiment, and each of the p columns gives a particular kind of feature (say, the results from a particular sensor).

Mathematically, the transformation is defined by a set of size ℓ of p-dimensional vectors of weights or coefficients $w^{(k)} = (w_1, \cdots, w_p)^{(k)}$ that map each row vector $x^{(i)}$ of X to a new vector of principal component scores

$$\mathbf{t}_{(i)} = (t_1, \dots, t_l)_{(i)},$$

given by

$$t_{k(i)} = \mathbf{x}_{(i)} \cdot \mathbf{w}_{(k)} \qquad \text{for} \qquad i = 1, \dots, n \qquad k = 1, \dots, \ell$$

Since $w^{(1)}$ has been defined to be a unit vector, it equivalently also satisfies

$$\mathbf{w}_{(1)} = \arg\max \left\{ \frac{\mathbf{w}^T \mathbf{X^T X w}}{\mathbf{w}^T \mathbf{w}} \right\} \mathbf{w}_{(1)} = \arg\max \left\{ \frac{\mathbf{w}^T \mathbf{X^T X w}}{\mathbf{w}^T \mathbf{w}} \right\}$$

The quantity to be maximised can be recognised as a Rayleigh quotient. A standard result for a positive semidefinite matrix such as $X^T X$ is that the quotient's maximum possible value is the largest eigenvalue of the matrix, which occurs when w is the corresponding eigenvector.

With $w(1)$ found, the first principal component of a data vector $x(i)$ can then be given as a score $t_1(i) = x(i) \cdot w(1)$ in the transformed co-ordinates, or as the corresponding vector in the original variables, $(x(i) \cdot w(1))w(1)$.

7.2 Singular value decomposition

$$A = USV^t$$

where U and V are real orthogonal matrices on R^n and S is a diagonal matrix consisting of singular values of A.

We give two arguments for existence of singular value decomposition. First, since

$$A^t A = (USV^t)^t USV^t = VSU^t USV^t = VS^2 V^t,$$

$$AA^t = USV^t (USV^t)^y = USV^t VSU^t = US^2 U^t,$$

thus

U corresponds to the eigenvectors of AA^t,

V corresponds to eigenvectors of $A^t A$

$S^2 = \tilde{\Lambda}$=eigenvalues of $A^t A$ (AA^t)

Singular values are similar in that they can be described from **variational principles**. Consider a constraint optimization

$$\max \quad \sigma(u, v) = u^t M v \text{ subject to } |u| \le 1, \;\; 1|v| \le 1$$

By Lagrange multiplier theorem with

$$L(u, v, \lambda_1, \lambda_2) = \sigma(u, v) + \lambda_1(|u|^2 - 1) + \lambda_2 \left(|v|^2 - 1\right)$$

there exists (u_1, v_1) such that

$$Mv_1 = \lambda_1 u_1, \;\; M^t u_1 = \lambda_2 \, v_1, \;\; |u_1| = |v_1| = 1.$$

Multiplying the first equation from by u^t and multiplying the second equation from by v^t, we have

$$\sigma_1 = u_1^t M v_1 = \lambda_1 = \lambda_2$$

The same calculation performed on the orthogonal complement $\{u \in R^n : (u, u_1) = 0\} \times \{v \in R^m : (v, v_1) = 0\}$ and gives the next largest singular value of M, and so. That is, we obtain singular value triples (σ_i, u_i, v_i) such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ and $\{u_i\}$ is orthonormal i.e. $(u_i, u_j) = \delta_{i,j}$, $\{v_i\}$ is orthonormal i.e. $(v_i, v_j) = \delta_{i,j}$ and

$$Mv_i = \sigma_i u_i, \quad M^t u_i = \sigma_i\, v_i, \quad |u_i| = |v_i| = 1.$$

Thus, $M = USV^t$.

Application (Image compression)

A bitmap image is represented by a $864 \times 1,536$ matrix, call it A Compute svd decomposition of $A = USV^t$ and and use a truncated svd

$$\tilde{A} = \tilde{U}\tilde{S}\tilde{V}^t$$

where $\tilde{S} = diag(s_1, \cdots, s_t)$, $\tilde{U} = (\vec{u}_1, \cdots, \vec{u}_t)$, $\tilde{V} = (\vec{v}_1, \cdots, \vec{v}_t)$. We select t such that the first t singular values of A dominate the remains singular values. It can be proved that $\tilde{A}$ is the optimal rank t approximation of A

$$|A - \tilde{A}|_F \text{ is smallest.}$$